The role of central bank digital currency in an increasingly digital economy*

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Abstract

The introduction of an unremunerated retail central bank digital currency (CBDC) is currently under consideration by several central banks. Motivated by the decline in transactional cash usage and the increase in online sales in the UK, this paper provides a theoretical framework to study the underlying drivers of these trends and the welfare implications of introducing an unremunerated retail CBDC. I develop a cash credit model with physical and digital retail sectors, endogenous entry of firms and directed consumer search. Calibrating to UK data between 2010 and 2022 the model suggests that there are positive welfare gains from introducing an unremunerated retail CBDC, but these have likely declined over time.

Keywords: CBDC, Credit, Digital Currency, Money

JEL codes: E41, E42, E58

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1 Introduction

The introduction of a retail central bank digital currency (CBDC) is currently being considered by the major central banks. Current proposals by the Bank of England and the ECB have suggested that should they decide to issue a CBDC, it would be unremunerated. This limits a lot of the financial stability concerns of issuing a CBDC, but also limits the potential benefits. An obvious question is if CBDC offers the same rate of return as cash, how does it differ from cash, and what is the motivation for introducing a CBDC?

Central banks offer several rationales for the introduction of a retail CBDC, with the emphasis varying across jurisdictions. One key rationale is to ensure the continued availability and utility of central bank money in a modern, digital economy. Additionally, a CBDC can enhance financial inclusion by providing access to digital financial services for unbanked or underbanked populations. Furthermore, a CBDC has the potential to foster innovation and competition in the payments sector, which could reduce costs and improve efficiency for consumers and businesses.

This paper focuses on two empirical trends that characterise a shift towards digital retail. The first is the decline in cash usage and corresponding increase in card payments in retail transactions. Figure 1 plots data from UK Finance's UK Payment Markets Report showing the number of payments made using cash, debit cards, and credit cards. The figure shows that the number of transactions in the UK using cash has decreased significantly since 2008. The second trend is that an increasing number of retail transactions are taking place online, where cash cannot typically be used as a means of payment. Figure 2 shows the share of internet sales as a percentage of total retail sales in the UK between January 2008 and July 2023. The chart captures the rapid increase in the share of online transactions from less than 5% to more than 25%. Even looking through a spike in the proportion of retail sales during the COVID-19 pandemic period, the series shows a steady upward trend.

However, it is not immediately clear whether the increase in online sales and decline in cash usage observed in the data strengthens the case for the issuance of a retail CBDC as a digital fiat money. The underlying drivers of this trend are likely to be an important factor in evaluating the benefits of issuing a CBDC. One possible scenario is that technological improvements in online retail are the primary driver of the shift to online sales and precipitating a decline in cash usage. Here, the introduction of a retail CBDC is likely to lead to significant welfare gains should it improve competition and efficiency in digital payments. An alternative scenario is that the decline in cash use and the increase in online sales is driven by improvements in digital payments technology. In this case, the introduction of a retail CBDC may have less pronounced welfare gains as the private sector is already providing much of the possible benefits.

The main focus of this paper is to understand the drivers behind the decline in cash usage and the increase in online sales and the implications of this for the introduction of a retail CBDC. This paper builds a theoretical framework that is able to model both the decline in cash usage and the increase in online retail sales while suggesting a mechanism through which the issuance of a CBDC could lead to aggregate welfare gains. Importantly, the model allows for several possible drivers of these trends with conflicting welfare implications for the introduction of a CBDC. To provide evidence on which of these drivers is most empirically consistent, the model is calibrated to UK data between 2010 and 2022.

The model developed in this paper is a cash-credit monetary model based on the framework developed by Lagos and Wright (2005) and Rocheteau and Wright (2005). Specifically, it builds on the recent paper by Lagos and Zhang (2022) who showed that the moneyless limit differs from a non-monetary model, as money acts as a constraint on the market power of financial intermediaries by acting as an outside option. This paper adopts this mechanism but extends the model in three key ways. First, I assume that there are two distinct firm types, digital and physical, which differ in the means of payment they are able to accept. Money is usable only in physical, face-to-face transactions and not in digital transactions. Second, I allow for endogenous firm entry into these two sectors, which is made possible by introducing search frictions and solving for the competitive search equilibrium. Allowing for endogenous firm entry allows the model to study different possible channels for the increasing trend in online sales identified by Figure 2 and the different welfare implications they have for the introduction of a CBDC. Finally, I allow consumers to choose between searching in the digital sector and searching in the physical sector. Consumers thus respond endogenously to changes in the relative attractiveness of the two sectors. For a detailed review of the competitive search equilibrium used in this paper, see Wright et al. (2021).

This paper has two key results. First, the paper finds that there are welfare gains from the introduction of CBDC. The welfare gains occur because, in the absence of a CBDC, the model suggests that the entry of online retailers is inefficiently low. This stems from physical retailers having a competitive advantage: their ability to accept cash, which constrains the market power of financial intermediaries and lowers the intermediation fees that physical retailers are required to pay to intermediaries. As online retailers are unable to accept cash as a means of payment, they face larger intermediation fees in order to access financial markets than physical retailers. Introducing a CBDC levels the playing field in the sense that online retailers have the same outside option as physical retailers, leading to efficient entry by both online retailers and physical retailers. Second,

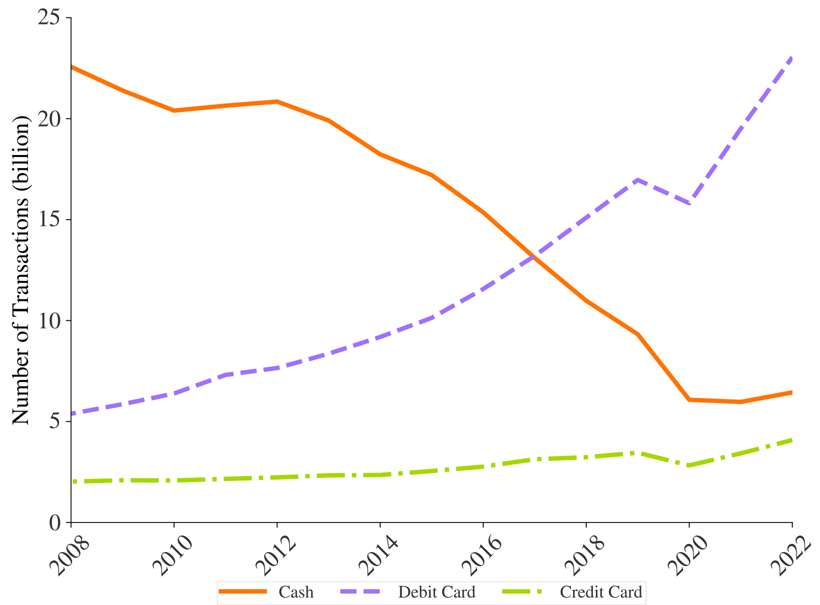

Figure 1: Total Number of UK Payments (in billions) by Method of Payment.
Source: UK Finance, UK Payment Markets Report. `https://www.ukfinance.org.uk`.

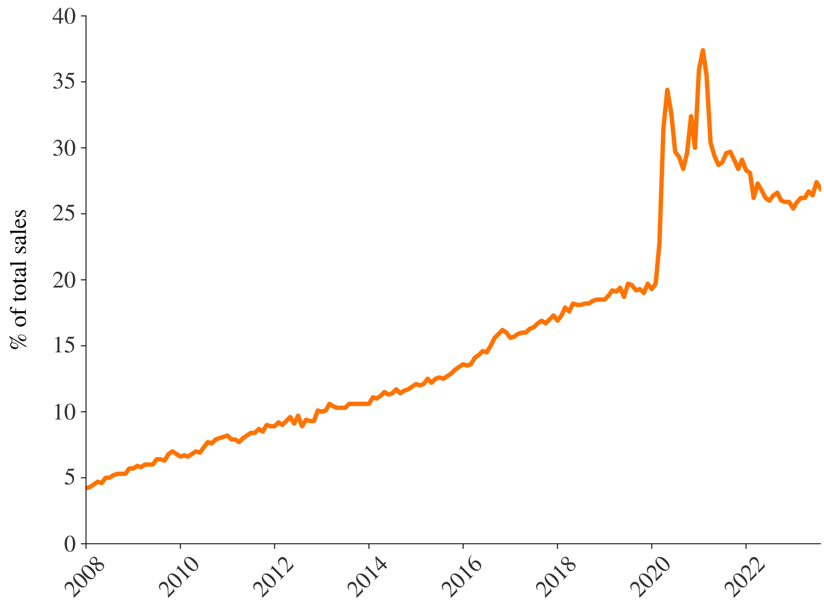

Figure 2: Value of Internet Sales as Percentage of Total Value of UK Retail Sales (seasonally adjusted).
Source: UK ONS, Retail Sales Index.

the paper finds that the welfare gains of introducing a CBDC have declined between 2010 and 2022. Calibrating the model to UK data suggests that the fall in cash usage and rise in online sales are driven primarily by improvements in digital payments rather than productivity increases in digital retail. As a consequence, the benefits of introducing a CBDC are lower.

There is extensive literature that discusses the welfare implications of introducing a CBDC, although, as pointed out by Bindseil and Senner (2024), much of this focusses on a remunerated CBDC, where CBDC offers a rate of return higher than cash, rather than an unremunerated CBDC which is more cash-like. For example, Barrdear and Kumhof (2022) find that a countercyclical remuneration rate rule for CBDC can contribute to stabilising the business cycle, while Bordo (2021) suggests that a remunerated CBDC can strengthen the transmission of monetary policy. This paper emphasises a possible benefit of an unremunerated CBDC, which can be seen as complementary to papers that highlight the potential risks of introducing a CBDC such as Fernández-Villaverde et al. (2021) that consider the increased risk of bank runs, especially in cases where the CBDC is remunerated.

Other papers such as Ferrari Minesso et al. (2022), Abad et al. (2024) and Bidder et al. (2024) study unremunerated CBDC by assuming that a CBDC is an imperfect substitute for other forms of money, such as cash and bank deposits. In these papers, the substitutability between different forms of money is fixed. In this paper, the substitutability between different forms of money depends on the ratio of digital and physical firms in the economy, which is endogenous. Although this paper focusses on digital transactions as a way to differentiate between CBDC and cash, there are other possible alternatives. For example, Burlon et al. (2022) differentiates between cash and CBDC by assuming that cash has higher storage costs than a potential CBDC.

Several recent papers have studied the introduction of a CBDC using money search models in the style of Lagos and Wright (2005). An example is Assenmacher et al. (2023) who study the effects of a remunerated CBDC in a New Keynesian New Monetarist model along the lines of Aruoba and Schorfheide (2011). Unlike this paper, they do not distinguish between physical money (cash) and digital money (CBDC). Williamson (2022) uses a similar money search framework to this paper while cash and CBDC are differentiated by the level of privacy they offer. Closer to this paper is the recent work of Jiang and Zhu (2021), Chiu et al. (2023), and Keister and Sanches (2023) who study the impact of a CBDC using money-search models that have distinct types of sellers that can accept different methods of payment. Keister and Sanches (2023) consider CBDC's substitutability with other means of payment as a design choice. Specifically, they distinguish between a cash-like CBDC that only competes with cash based payments, a deposit-like CBDC which competes with bank deposits, and a universal CBDC which competes with bank deposits and cash payments. A key modelling difference in this paper is that search frictions are introduced and allow for both endogenous entry of firms and consumers into the digital and physical markets. This allows the model to study the underlying drivers behind the decline in cash usage and the increase in online retail.

This is not the first paper to consider the impact of a declining trend in cash use. Jiang and Shao (2020) not only document the decline in cash usage, but also suggest an answer to the 'cash paradox', that is why cash usage has not declined as much as might be expected. Khiaonarong and Humphrey (2022) suggest that the decline in cash usage is linked to a generational shift in payment preferences and explore the implications of this trend for CBDC adoption. A paper closer in spirit to this one is Chiu et al. (2023), who study the impact of an exogenous shift toward sellers that do not accept cash. As I allow for endogenous entry of firms, this paper is able to show that the welfare implications of introducing a CBDC may depend on the underlying factors driving the declining trend in cash usage. Similarly Panetta (2021) refers to as the role of central bank money as monetary anchor and the role a retail CBDC may have in a cashless society.

The remainder of this paper is organised as follows. Section 2 presents the model. Section 3 describes the equilibrium and discusses the impact of introducing a CBDC. Section 4 calibrates the model to UK data and studies the effect of introducing a CBDC. Section 5 concludes.

2 The Model

In this section, I build a cash-credit model based on the framework of Lagos and Zhang (2022) with two innovations. First, the model features two distinct types of retailer, a physical retailer and a digital retailer. The retailers are distinguished in the means of payment they are able to accept. Physical retailers are able to accept cash as a means of payment, whereas digital retailers cannot. Second, consumers are subject to search frictions and choose between physical and digital retailers in a directed search setting along the lines of Rocheteau and Wright (2005).

2.1 Environment

Time is discrete, lasts forever, and is represented by a sequence indexed by $t \in \mathbb{T} \equiv \{0, 1, \dots\}$. There are three types of agents, denoted by $i \in \{c, f, b\}$; a measure 1 of consumers, c, a measure 1 of financial intermediaries b, and firms, f. Firms are divided into two permanent types: physical retailers and digital retailers, denoted by p and d, respectively. There is a measure $\bar{N}$ of potential firms that choose which sector to enter at the beginning of t subject to an entry cost. Consumers and financial intermediaries live forever, while firms live only for one period.

The discount factor from the current period to the next is $\beta \in (0,1)$. Each period is divided into two stages: first, a decentralised frictional trading stage (DM), and second, a centralised frictionless settlement stage (CM). There are two nonstorable goods: y in the DM and x in the CM.

At the beginning of period t there is a quantity M_t of money. Money is an intrinsically useless financial asset issued by the monetary authority. Money is perfectly divisible, and agents can hold any non-negative amount. In the baseline model, money is physical and can only be used as a medium of exchange by physical retailers. In this sense, money can be thought of as being banknotes that cannot be used in transactions with online retailers. Later, I consider the possibility that the monetary authority issues digital cash, such as an unremunerated central bank digital currency (CBDC) that can be used by both digital retailers and physical retailers. The initial money stock, $M_0 \in \mathbb{R}_{++}$, is taken as given and distributed uniformly among consumers. The monetary authority is assumed to constantly adjust the money supply through lump-sum transfers or taxes to consumers in the CM stage of every period so that the law of motion of the money supply is $M_{t+1} = \mu M_t$ with $\mu \in \mathbb{R}_{++}$.

Also, at the beginning of period t, firms choose which sector $j \in \{p, d\}$ to enter subject to a free entry condition. I assume that firms face a cost of entering a sector $\eta\left(N_{f,j}\right)$ is increasing in the mass of firms that enter each sector, $N_{f,j}$, with $\eta' > 0$, $\eta'' \geq 0$ and $\eta(0) = 0$. I assume that the mass of potential entrants is large enough that in equilibrium $\sum_j N_{f,j} \ll \bar{N}$ and the entry of firms drives the expected profits in both sectors to zero. Firms that enter a sector at the beginning of the period begin with no assets and fund their cost of entry with bonds issued to a financial intermediary which they commit to repay through x in the CM.

In the first stage, consumers obtain utility from consuming y of the DM good which can only be produced by firms. The utility consumers get from consumption in the DM is $u(y)$, with $u'(0) = \infty$, $u' > 0$, $u'' < 0$ and $u(0) = 0$. Firm j's marginal cost of producing y , which may depend on their type, is denoted by $\kappa_j > 0$.

In the second stage, agents of all types consume the CM good, x, and are able to supply labour, h, which can be used to produce good x through a linear production technology. All agents obtain utility $v_i(x)$ from consuming x of the CM good, with $v_i'(0) = \infty$, $v_i' > 0$,

$v_i'' < 0$ and there exists $x_i^* > 0$ such that $v_i'(x_i^*) = 1$. In order to simplify the firm's entry problem, it is assumed that $v_f(x_i^*) = x_i^*$ so that in equilibrium firm utility in the CM is normalised to zero.

As is common in money-search models, money has a meaningful role as a medium of exchange because consumers are unable to commit in the DM and firms cannot enforce consumers' promises. Financial intermediaries are endowed with the ability to enforce and commit, and thus are able to play the role of financial intermediaries between consumers and producers. Specifically, as in Lagos and Zhang (2022), consumers obtain credit from financial intermediaries in the first stage in order to purchase goods in the DM. A credit contract allows consumers to purchase goods in the DM in exchange for a claim on the CM good, essentially a consumer issued bond. These credit contracts can be thought of as being credit card payments where I have abstracted from issues surrounding default.

In the second stage, all agents can trade the CM good and money in a spot Walrasian market. In the first stage, there are two markets: a goods market where money and bonds are exchanged for the DM good and a financial market where money and bonds are traded.

The financial market in the first stage is organised as follows. All financial intermediaries have access to the financial market, where they can trade bonds and money competitively. All consumers are able to access the financial market through a randomly assigned financial intermediary. Consumers are assumed to make a take-it-or-leave-it offer to the financial intermediary. Only a random subset of firms are able to access the financial market. With probability $\alpha \in [0, 1]$, a firm matches bilaterally with a financial intermediary.

If a firm and an intermediary make contact, they are able to accept bonds as a means of payment in exchange for a fee paid to the financial intermediary. The intermediation fee is expressed in terms of the CM good and paid in the second stage. The intermediation fee is determined by Nash bargaining between the firm and intermediary, where the firm has bargaining power $\theta \in [0, 1]$. I assume that firms that do not have access to financial markets through an intermediary are unable to accept credit payments. Thus the intermediation fee captures the real world card processing fees that firms pay to merchant acquirers. The assumption that intermediaries have market power when dealing with firms but not with consumers captures the free-banking system in the UK where banking payments are free for consumers but costly for merchants.

As the intermediation fee is set through Nash bargaining, firms that are able to access financial markets will always be strictly better off. The parameter α captures the fact that some physical retail firms do not accept card payments. As α is realised after firms enter the submarket, there are two effects on digital firms. First, as some firms pay the entry cost and are unable to accept payments, a lower α raises the effective entry cost of physical firms. In addition, as these firms still enter the submarket, a lower α reduces the match efficiency in the digital submarket. This captures the difficulty online retailers may have in establishing themselves in an economy where digital payments are less pervasive. The UK Payment Systems Regulator (PSR) in their Market Review into card-acquiring services (Payment Systems Regulator, 2021) note that card acquirers must perform due diligence and be comfortable in the risk (both credit risk and other forms of risk) that onboarding a new firm entails. The PSR also notes that some card acquirers choose not to serve entire industries that they deem too risky. Improvements in payment technology over time is likely to have expanded the availability of card payments to a larger range of merchants.

The goods market in the first stage is organised according to the competitive search equilibrium, as described in Rocheteau and Wright (2005). There is a price posting mechanism where the terms of trade are publicly announced. As in Moen (1997), this can be interpreted as arising from competition between market makers. Trade takes place bilaterally between firms and consumers, and each submarket is subject to trading frictions that occur due to a search externality. The set of open submarkets is partitioned into two subsets indexed by firm types $j \in \{p, d\}$. Consumers can choose to enter any submarket from the set of open submarkets, while firms are only able to enter open submarkets in the subset of open submarkets indexed by their firm type. Formally, consider an open submarket for type j firms indexed by j_s. The ratio of firms to consumers in submarket j_s is denoted by $n_{j_s} = n_{f,j_s}/n_{c,j_s}$ where $n_{f,j_s} \leq N_{f,j}$ is the mass of firms that enter submarket j_s from the mass of entrants into sector j, $N_{f,j}$, and n_{c,j_s} is the mass of consumers that enter submarket j_s.

The probability that a consumer who searches in submarket j_s matches with a firm is $\delta\left(n_{j_s}\right)$. As trade occurs bilaterally, the probability that a firm matches with a consumer is $\delta\left(n_{j_s}\right)/n_{j_s}$. It is assumed that $\delta'\left(n\right) > 0$, $\delta''\left(n\right) < 0$, $\delta\left(n\right) \leq \min\{1, n\}$, $\delta\left(0\right) = 0$, $\delta'\left(0\right) = 1$, and $\delta\left(\infty\right) = 1$. The market maker posts the terms of trade before the firm knows if it has access to financial markets. Each submarket j_s consists of the following $\omega_{j_s} = \left(y_{j_s}, p_{j_s}, n_{j_s}\right)$. Where n_{j_s} is the ratio of firms to consumers in the submarket and y_{j_s} and p_{j_s} are the quantity and nominal price traded conditional on the ability of the firm and the consumer to trade. In the physical sector, matched consumers and firms are always able to trade. Physical retailers with access to the financial market can trade using cash or card payments, while physical retailers without access to intermediaries trade using money. In the baseline model, digital retailers cannot trade using money, and thus matched consumers and firms in the digital sector can only trade if the firms obtain access to the financial market and are able to process card payments. Figure 3 sets out

	t				
Firms	Firms choose sector $j \in \{p, d\}$ to enter	Enter submarket $j_s \in \Sigma_j$ with (x_{j_s}, φ_{j_s})	Match with consumers with probability $\frac{\delta(n_{j_s})}{n_{j_s}}$. Match with intermediary with probability α and negotiate fee k	Sell x_{j_s} at price φ_{j_s} conditional on acceptable medium of exchange	Produce, trade, consume y and exit
	t				
Consumers	Begin period with money holdings m	Enter submarket $j_s \in \Sigma$ with (x_{j_s}, φ_{j_s})	Match with firms with probability $\delta(n_{j_s})$	Buy x_{j_s} at price φ_{j_s} conditional on acceptable medium of exchange	Produce, trade, consume y and choose future money holding m'
	t				
Financial Intermediaries	Begin period with money holdings m		Match with firms with probability α and negotiate fee k		Produce, trade, consume y and choose future money holding m'

Figure 3: Model Timing

the model timing for each of the three agent types.

The instantaneous utility of a consumer at date t is

$$U_{c,t} = u(y_t) + v(x_t) - h_t \tag{1}$$

and a consumer's lifetime utility is $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U_{c,t}$. Similarly, the instantaneous utility of a financial intermediary at date t is

$$U_{b,j,t} = v(x_t) - h_t \tag{2}$$

and the financial intermediary's lifetime utility is $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U_{b,t}$. Finally, firms live for a single period and the utility of a type $j \in \{p, d\}$ firm that is active in date t is

$$U_{f,j,t} = -\kappa_j y_t + v(x_t) - h_t. \tag{3}$$

2.2 Efficient Allocation

A useful benchmark to consider is the efficient allocation. Given that the matching function $\delta(\cdot)$ is concave, it is optimal for the social planner to create one submarket per sector. Thus, we can consider a social planner that chooses n_j as the ratio of firm type $j \in \{p, d\}$ to consumers in each sector, the mass of firms in each sector $N_{f,j}$, as well as an allocation $\Lambda = \left\{ (y_{j,t})_{j\in\{p,d\}}, (x_{i,t}, h_{i,t})_{i\in\{c,f,b\}} \right\}_{t=0}^{\infty}$. The planner maximises the equally weighted utility of all agents at each point in time t. Thus, let

$$\begin{aligned}\mathcal{W} = & \sum_{j\in\{p,d\}} \left(\frac{n_{f,j}}{n_j}\delta\left(n_j\right)\left[u\left(y_{c,j,t}\right) - \kappa_j y_{f,j,t}\right] - N_{f,j}\eta\left(N_{f,j}\right)\right) \\ & + \sum_{i\in\{c,b\}} \left(v_i\left(x_{i,t}\right) - h_{i,t}\right) + \sum_{j\in\{p,d\}} N_{f,j}\left(v_f\left(x_{f,t}\right) - h_{f,t}\right).\end{aligned} \tag{4}$$

where $n_{f,j}$ is the mass of firms of type j that attempt to match with consumers, $N_{f,j}$ is the mass of firms that enter sector j and $n_j = n_{f,j}/n_{c,j}$ is the ratio of firms of type j to the number of consumers who want to purchase from firms of type j. The planner aims to maximise $\sum_{t=0}^{\infty}\beta^t\mathcal{W}_t$ subject to several feasibility constraints. First, the allocation in the DM goods market must be feasible $y_{c,j,t} \leq y_{f,j,t}$. Similarly, the allocation in the CM goods market must also be feasible, $\sum_{i\in\{c,f,b\}} x_{i,t} \leq \sum_{j\in\{c,f,b\}} h_{i,t}$. Next, the mass of firms wishing to trade must be feasible, and thus $n_{f,j} \leq N_{f,j}$ must hold for all $j \in \{p,d\}$. Finally, since consumers must trade with at most one type of firm, $\sum_j n_{c,j} \leq 1$. Thus, an efficient allocation is an allocation Λ that maximises $\sum_{t=0}^{\infty}\beta^t\mathcal{W}_t$ subject to the feasibility constraints.

The efficient allocation consists of $y^*_{f,j,t} = y^*_{c,j,t} = y^*_{j,t}$ where $y^*_{j,t} = u'^{-1}\left(\kappa_j\right)$ and $x^*_i = v'^{-1}(1)$ for all $i \in \{c,f,b\}$. Turning to the efficient submarket composition, the first order condition from the planner's problem yields

$$\delta'\left(n_j\right)S^*_j = \eta\left(N_{f,j}\right) + N_{f,j}\eta'\left(N_{f,j}\right), \tag{5}$$

where $S^*_j \equiv u\left(y^*_{j,t}\right) - \kappa_j y^*_{j,t}$ denotes the surplus of trading with a firm in sector $j \in \{p,d\}$. As firm entry is costly, it is efficient for all firms that enter a given sector to also enter the submarket, and thus $n_{f,j} = N_{f,j}$ for all $j \in \{p,d\}$.

An efficient allocation where both digital and physical firms enter requires that all consumers attempt to trade, $\sum_j n_{c,j} = 1$, and that the following condition holds

$$\left(\delta\left(n_d\right) - n_d\delta'\left(n_d\right)\right)S^*_d = \left(\delta\left(n_p\right) - n_p\delta'\left(n_p\right)\right)S^*_p. \tag{6}$$

Equation (6) highlights the trade-off between sectors. A sector with a higher surplus will be offset in equilibrium by a lower n and thus more congestion.

For a solution to equation (6) to exist, it must be the case that the ratio of S^*_d to S^*_p is not too large or too small. In the case where S^*_d is significantly larger than S^*_p, it is optimal for consumers to trade only with digital firms, and thus no physical firms would enter the market. As all firms enter the digital sector, $n_{c,d} = 1$ and thus $n_d = N_{f,d}$. From the properties of $\delta(\cdot)$, it follows that $\lim_{n_j\to 0}\left(\delta\left(n_j\right) - n_j\delta'\left(n_j\right)\right) = 1$. It follows that there is no solution to equation (6) and that this corner solution is optimal if

$$\delta\left(N_{f,d}\right)-\delta'\left(N_{f,d}\right)N_{f,d}>\frac{S_p^*}{S_d^*} \tag{7}$$

where $N_{f,d}^*$ can be found as the solution to $\delta'\left(N_{f,d}^*\right)S_d^*=\eta\left(N_{f,d}^*\right)+N_{f,d}^*\eta'\left(N_{f,d}^*\right)$.

Similarly, the converse would hold in the case where S_p^* is significantly larger than S_d^* such that

$$\delta\left(N_{f,p}^*\right)-\delta'\left(N_{f,p}^*\right)N_{f,p}^*>\frac{S_d^*}{S_p^*} \tag{8}$$

where $N_{f,p}^*$ can be found as the solution to $\delta'\left(N_{f,p}^*\right)S_p^*=\eta\left(N_{f,p}^*\right)+N_{f,p}^*\eta'\left(N_{f,p}^*\right)$. Here, all consumers attempt to trade with physical firms $n_{c,p}=1$, and no firms would enter the digital submarket.

2.3 Settlement in the centralised market

Consider the utility of an agent of type $i\in\{c,f,b\}$ who enters the second stage with money holdings m and bond holdings a. Their value function can be expressed as

$$\begin{aligned} W_i\left(m,a\right) &= \max_{x,h,m'}\left[v\left(x\right)-h+\beta V_i\left(m'\right)\right], \\ \text{s.t.} \qquad x+\phi m' &\leq h+a+\phi\left(m+\mathbb{I}_{\{i=c\}}T\right), \end{aligned} \tag{9}$$

where $T\in\mathbb{R}$ is the lump-sum monetary injection to an individual consumer, and ϕ is the value of one unit of money in terms of the CM good x which is the numeraire.

As a welfare maximising agent will ensure their budget constraint binds, by substituting out h_t using the budget constraint equation (9) can be written as

$$W_i\left(m,a\right)=\phi m+a+\bar{W}_i, \tag{10}$$

where

$$\bar{W}_i\equiv\mathbb{I}_{i=c}\phi T+v\left(x^*\right)-x^*+\max_{m'}\left[\beta V_i\left(m'\right)-\phi m'\right]. \tag{11}$$

The first-order condition with respect to the money demand of an agent of type $i\in\{c,b\}$ yields the following Euler equation

$$\beta\frac{\partial V_i\left(m'\right)}{\partial m'}\leq\phi, \quad \text{with `` $=$ '' if } m'>0 \text{ for } i\in\{c,b\}. \tag{12}$$

2.4 Portfolio reallocation

For an agent who has access to the financial market at the end of the first subperiod, they are able to adjust their portfolio following any transactions made before entering the second subperiod. Consider the problem of an agent that ends the first subperiod with a quantity $\tilde{m}$ of money and $\tilde{a}$ of bonds in the form of claims on good x in the second subperiod. Agents with access to the financial market are able to trade money and bonds, where one unit of money purchases $\frac{1}{q}$ bonds. The portfolio reallocation problem is for agents to choose bond holdings a subject to the budget constraint $m + qa \leq \phi\tilde{m} + q\tilde{a}$. This yields the following problem

$$\max_{a} W_i\left(\tilde{m} - q\left(a - \tilde{a}\right), a\right) = \phi\left(\tilde{m} - q\left(a - \tilde{a}\right)\right) + a + \bar{W}_i, \tag{13}$$

subject to agents holding a non-negative quantity of money, $m = \tilde{m} - q\left(a - \tilde{a}\right) \geq 0$.

It is useful to define the interest rate on bonds as i where $(1+i) = \frac{1}{\phi q}$. In equilibrium, the interest rate on bonds will be pinned down by the monetary authority through the growth rate of money μ. I assume that monetary policy is set so that $\mu > \beta$ and thus the nominal interest rate will be strictly positive $i > 0$. Given this equation (13) can be written as

$$\max_{a} W_i\left(\tilde{m} - qa, a\right) = \phi\tilde{m} + \frac{i}{1+i}a + \frac{1}{1+i}\tilde{a} + \bar{W}_i, \tag{14}$$

and as $i > 0$ it is optimal for agents who have access to financial markets not to carry money into the second subperiod. As a consequence, only firms that are unable to access financial markets would hold money between the subperiods.

2.5 Trade in the Decentralised market

A key distinction between sectors is their ability to accept cash as a means of payment in the decentralised market. I introduce the dummy variable γ_j that takes the value 1 if the firms in sector j are able to accept cash as a means of payment and 0 if not. Given this, in the benchmark model, where only physical firms are able to accept cash, γ_j takes the following values

$$\gamma_j = \begin{cases} 1 & \text{if } j = p \\ 0 & \text{if } j = d. \end{cases} \tag{15}$$

Consider the choice of a consumer who begins the period with money holdings m and chooses to enter submarket j_s. The submarket j_s consists of posted terms of trade $\omega_{j_s} = (y_{j_s}, \varphi_{j_s})$ and in equilibrium attracts a ratio of firms to consumers of n_{j_s}. Where y_{j_s} is the quantity of the good posted and φ_{j_s} is the posted price in terms of the numeraire good x. The price of the good in terms of money is thus $\frac{\varphi_{j_s}}{\phi}$. The ratio of firms to consumers in submarket j_s is n_{j_s}. The consumer's value function can be written as

$$\begin{aligned} V_c(m) = \max_{\omega_{j_s}\in\Omega} \Big\{ \delta(n_{j_s})\left[\alpha + \gamma_j(1-\alpha)\right] \left[u(y_{j_s}) + \max_a W_c\left(m - \frac{1}{\phi}\varphi_{j_s} y_{j_s} - qa, a\right)\right] \\ + \left(1 - \delta(n_{j_s})\left[\alpha + \gamma_j(1-\alpha)\right]\right) \max_a W_c(m - qa, a) \Big\}. \end{aligned} \tag{16}$$

With probability $\delta(n_{j_s})$, the consumer matches with a firm. Conditional on matching with a firm trade will occur with probability $\alpha + \gamma_j(1-\alpha)$ which is the probability an acceptable means of payment can be found. Thus, in the physical sector with $\gamma_p = 1$ matched consumers and firms will always trade, while in the digital sector where $\gamma_d = 0$, matched consumers and firms only trade with probability α where firms have access to financial markets that allow them to process card payments. As consumers are able to rebalance their portfolios irrespective of whether they trade with a firm, the continuation value of a consumer that does not match with a firm is the same as the continuation value of a consumer that matches with a firm but is unable to trade with them.

As all consumers have access to financial markets, equation (16) can be rewritten by using the solution to the optimal portfolio reallocation set out by equation (14) such that

$$V_c(m) = \max_{\omega_{j_s}\in\Omega} \Big\{ \delta(n_{j_s})\left[\alpha + \gamma_j(1-\alpha)\right]\left[u(y_{j_s}) - (1+i)\varphi_{j_s} y_{j_s}\right] + (1+i)\phi m + \bar{W}_c \Big\}. \tag{17}$$

Turning now to the firm problem, firms live for one period and enter the period with a debt equal to the cost of entry, which is payable in the second subperiod in terms of the CM good. To characterise the firm's value function, I first consider the value of a firm at the end of the first sub-period that enters sector j and matches with a firm in submarket j_s.

In the case where the firm cannot access financial markets, the firm is unable to adjust its asset portfolio at the end of the first subperiod and is only able to trade if it can accept cash ($\gamma_j = 1$). Thus, the firm ends the subperiod with a quantity of money equal to $\gamma_j \frac{1}{\phi}\varphi_{j_s} y_{j_s}$ and holdings of bonds equal to $-\eta(N_{f,j})$. Thus the value of the firm at the start of the second subperiod is

$$W_f\left(\gamma_j \frac{1}{\phi}\varphi_{j_s} y_{j_s}, -\eta(N_{f,j})\right) = \gamma_j \varphi_{j_s} y_{j_s} - \eta(N_{f,j}) \tag{18}$$

where the above equation uses the fact that the firm's CM good value function is defined such that $v_f(x^*) - x^* = 0$ and thus $\bar{W}_f = 0$.

With probability α, a firm that matches with a consumer negotiates a fixed fee k with the financial intermediary in order to access the financial market. Firms and intermediaries set the fee k through Nash bargaining and solve the following problem

$$\begin{aligned} &\max_{k,\bar{m},a} \left[W_f\left(\bar{m},a\right) - W_f\left(\gamma_j \frac{1}{\phi}\varphi_{j_s} y_{j_s}, -\eta\left(N_{f,j}\right)\right)\right]^{\theta} k^{1-\theta}, \\ \text{s.t.} \qquad &\phi \bar{m} \leq \varphi_{j_s} y_{j_s} - \left(\frac{1}{1+i}\right)\left(a + k + \eta\left(N_{f,j}\right)\right), \\ &W_f\left(\gamma_j \frac{1}{\phi}\varphi_{j_s} y_{j_s}, -\eta\left(N_{f,j}\right)\right) \leq W_f\left(\bar{m},a\right). \end{aligned} \tag{19}$$

The firms benefit from access to financial markets in two ways. First, firms are able to accept card payments from consumers and thus matched firms will be able to trade regardless of their sector. Second, firms are able to rebalance their portfolio before entering the second subperiod. As set out in equation (14), firms with access to financial markets will choose not to hold cash between the two subperiods, benefiting from the positive interest rate on bonds.

The surplus generated by allowing firms to access the financial market is

$$\Delta W_{f,j} = \left(1 + i - \gamma_j\right)\varphi_{j_s} y_{j_s} - k \tag{20}$$

and the solution to the Nash bargaining problem yields the following

$$k = \left(1 - \theta\right)\left(1 + i - \gamma_j\right)\varphi_{j_s} y_{j_s}, \tag{21}$$

where the intermediation fee is set such that it captures a share $(1-\theta)$ of the total surplus. A key determinant of the surplus and hence the fee is the outside option of the firm. As firms that are able to accept cash as a means of payment have a larger outside option, and hence face lower intermediation fees. It follows that the model predicts that online retailers are likely to pay higher card processing fees than physical stores. This matches the experience in the UK where the PSR notes that merchants tend to pay higher fees on transactions where cardholders are not physically present (Payment Systems Regulator, 2021). They also note that some merchants pay additional fees for e-commerce transactions.

At the beginning of the period, nascent firms choose which sector to enter. The value of a firm entering sector j at the beginning of a period depends on $N_{f,j}$, the total mass of entrants into that sector. Given the above discussion, this value is given by the following equation

$$V_{f,j}\left(N_{f,j}\right)=\max_{\omega_{j_s}\in\Omega_j}\left\{\frac{\delta\left(n_{j_s}\right)}{n_{j_s}}\left[\left(1+\alpha\theta i\right)\varphi_{j_s}y_{j_s}-\kappa_j y_{j_s}\right]\right.$$
$$\left.-\left(1-\gamma_j\right)\frac{\delta\left(n_{j_s}\right)}{n_{j_s}}\left[\left(1-\alpha\theta\right)\varphi_{j_s}y_{j_s}-\left(1-\alpha\right)\kappa_j y_{j_s}\right]-\eta\left(N_{f,j}\right)\right\}, \tag{22}$$

where firms in sector j are limited in choosing a submarket ω_{j_s} from the partition of submarkets designed for sector j firms, $\Omega_j \subset \Omega$.

Finally, consider a financial intermediary that begins the period with money holdings m. The value function of a financial intermediary is

$$V_b\left(m\right)=\max_a W_b\left(m-q\left(a-\alpha\sum_{j_s}n_{c,j_s}\delta\left(n_{j_s}\right)k_{j_s}\right),a\right). \tag{23}$$

Using the fact that the fee revenue is $k_{j_s}=\left(1-\theta\right)\left(1+i-\gamma_j\right)\varphi_{j_s}y_{j_s}$, and solving for the optimal portfolio reallocation, this can be rewritten using equation (14) as

$$V_b\left(m\right)=\alpha\sum_{j_s}n_{c,j_s}\delta\left(n_{j_s}\right)\left(1-\theta\right)\left(1+i-\gamma_j\right)\varphi_{j_s}y_{j_s}+\phi\left(1+i\right)m+\bar{W}_b. \tag{24}$$

Financial intermediaries do not buy or sell goods in the first subperiod but instead collect intermediation fees from a fraction α of firms that match with consumers.

3 Equilibrium

In this section, I focus on characterising the steady-state equilibrium, which is defined as follows:

Definition 1. *A steady-state equilibrium is an interest rate i, real money balances $\phi_t M_t$, a set of open submarkets Ω, a partition $\{\Omega_p,\Omega_d\}$, for each submarket $\omega_j\in\Omega_j$ a list $\left(y_{\omega_j},\varphi_{\omega_j},n_{\omega_j},n_{c,\omega_j},k_{\omega_j}\right)$ and a mass of firms entering each sector $\left(N_{f,p},N_{f,d}\right)$ such that:*

- *i) taking prices as given, the end-of-period money holdings solve equation (12) for $i\in\{c,b\}$ and supply of bonds in the intra-period financial market clears with bonds in zero net supply;*
- *ii) the set of open submarkets maximises consumer welfare, equation (17) subject to $V_{f,j}\left(N_{f,j}\right)\geq 0$, $\sum_{\omega_j\in\Omega_j}n_{f,\omega_j}\leq N_{f,j}$ and $\sum_{\omega_j\in\Omega}n_{c,\omega_j}=1$;*
- *iii) intermediary fees solve the bargaining problem in equation (19);*
- *iii) firms entering sector $j\in\{p,d\}$ make zero profit in expectation;*

I focus on a monetary equilibrium where real money balances $\phi_t M_t > 0$ are constant over time. First, in equilibrium, the interest rate on intra-period bonds is $i = \mu/\beta - 1$. This equates the marginal cost of holding money, which depends on the growth rate of money (μ) with the marginal return of holding money in the next period decentralised market, $1 + i$.

Consider the next end-of-period money demand. This is characterised by equation (12) for $i \in \{c, b\}$, while firms by assumption live for only one period and thus are unable to hold money between periods. From equation (14) it follows that agents with access to financial markets would choose not to hold money between the first subperiod and second subperiod. Only firms without access to financial intermediaries and are able to accept money as a means of payment hold money between the two subperiods. Aggregate demand for real money balances is then

$$\phi_t M_t = (1 - \alpha) \sum_j \gamma_j n_{c,j} \delta(n_j) \varphi_j y_j. \tag{25}$$

Thus, a monetary equilibrium requires $\alpha < 1$, although, as in Lagos and Zhang (2022), the cashless limit where $\alpha \to 1$ can also be considered. Due to the concavity of the matching function, $\delta(\cdot)$, in equilibrium, there will be at most one submarket for each sector j. Thus, to simplify the notation, I now drop the subscript s, as in the above equation, and index the submarkets by sector $j \in \{p, d\}$.

The open submarket for sector j consists of a price φ_j, a quantity y_j, and a tightness $n_j = n_{f,j}/n_{c,j}$ that maximises equation (17) subject to $V_{f,j}(N_{f,j}) \geq 0$, taking $N_{f,j}$ as given.

Consumers are free to choose which of the submarkets to enter, and thus, for a positive mass of consumers to enter both submarkets in equilibrium, consumers must be indifferent between the two submarkets. With some abuse of notation, denote $V_{c,j}$ as the expected utility of a consumer who chooses to shop in submarket j. Thus, a requirement for both sectors to actively trade is $V_{c,p} = V_{c,d}$. In the case where $V_{c,p} \neq V_{c,d}$, all consumers search in only one of the sectors, leaving the other sector inactive. As there are gains from trade and searching in a submarket is costless, all firms enter a submarket $\sum_j n_{c,j} = 1$.

Given that there is free entry into each of the sectors, firms enter until the expected profits in each sector are driven to zero: $V_{f,j}(N_{f,j}) = 0$. As firms that have entered sector j commit to paying the entry cost and searching in a submarket is costless, any firm that chooses to enter the market will also attempt to trade, and thus $n_{f,j} = N_{f,j}$.

3.1 Equilibrium without CBDC

In this section, I further characterise the equilibrium in the benchmark model without CBDC. In this case, γ_j is defined as in equation (15) and only firms in the physical sector are able to accept cash as a means of payment.

Solving first for the posted quantity in both submarket yields

$$y_d = u'^{-1}\left(\frac{1}{\theta}\kappa_d\right) \tag{26}$$

and

$$y_p = u'^{-1}\left(\left(\frac{1+i}{1+\alpha\theta i}\right)\kappa_p\right), \tag{27}$$

for the digital and physical submarkets respectively.

Compared with the optimal consumption level, consumption in the digital sector is lower than optimal in the digital sector whenever $\theta < 1$, while consumption in the physical sector is lower than optimal whenever $i > 0$ or $\alpha\theta < 1$.

It is instructive to consider the case where $\theta = 1$ and $\alpha < 1$. Here, firms have imperfect access to the financial market, but access is costless ($k_j = 0$). In this case, while consumption in the digital sector is at the optimal level, trade between matched firms and consumers occurs only with probability α. Trade in the digital sector is conditional on access to the financial market, but the means of payment is costless. On the other hand, trade between matched firms and consumers will always take place, but the quantity traded will be lower than optimal so long as $i > 0$. The distortion in quantity traded occurs because trade without access to financial markets is possible but requires the use of cash, which is a costly means of payment if $i > 0$.

Taking the mass of firms entering each sector as fixed, the equilibrium tightness in sector j can be found by maximising equation (17) with respect to n_j subject to equation (22) being nonnegative. This yields

$$\delta'(n_d)\,\Gamma_d = \frac{1}{\alpha\theta}\eta\left(N_{f,d}\right), \tag{28}$$

and

$$\delta'(n_p)\,\Gamma_p = \left(\frac{1+i}{1+\alpha\theta i}\right)\eta\left(N_{f,p}\right), \tag{29}$$

for the digital and physical submarkets respectively. Here, $\Gamma_j \equiv u(y_j) - y_j u'(y_j)$ denotes the surplus available from trade between a consumer and a type j firm for a given consumption y_j.

Using the above first order conditions and the participation constraint of firms $V_{f,j}(N_{f,j}) = 0$, equation (17) yields the following equation for expected utility of consumers that attempt to trade in the digital sector submarket

$$V_{c,d} = \alpha \left(\delta \left(n_d\right) - n_d \delta' \left(n_d\right)\right) \Gamma_d, \tag{30}$$

and the following equation for consumers attempting to trade in the physical submarket

$$V_{c,p} = \left(\delta \left(n_p\right) - n_p \delta' \left(n_p\right)\right) \Gamma_p. \tag{31}$$

The above equations state that consumer utility is increasing in the ratio of firms to consumers in each submarket, n_j, and in the surplus from trade, Γ_j. In the digital sector, since money cannot be used as a means of payment, consumer utility is discounted by a factor α, the probability that firms are able to access financial markets.

The equilibrium prices, φ_j for $j \in \{p, d\}$ can be found from the corresponding firm participation constraints, $V_{f,j}(N_{f,j}) = 0$, taking firm entry as given. This yields

$$(1+i) \varphi_d y_d = y_d u' \left(y_d\right) + \frac{1}{\alpha\theta} \frac{n_d}{\delta \left(n_d\right)} \eta \left(N_{f,d}\right), \tag{32}$$

and

$$(1+i) \varphi_p y_p = y_p u' \left(y_p\right) + \left(\frac{1+i}{1+\alpha\theta i}\right) \frac{n_p}{\delta \left(n_p\right)} \eta \left(N_{f,p}\right), \tag{33}$$

for the digital and physical sectors respectively.

Similarly to the efficient allocation, if the surplus from trade in one sector is sufficiently high relative to the other sector, the equilibrium will lie at a corner where only one sector is active.

Consider first the case where consumers only trade in the digital sector and thus $n_{c,d} = 1$ and $n_d = N_{f,d}$. This is an equilibrium in the case where consumers strictly prefer to enter the digital submarket and thus $V_{c,d} > V_{c,p}$. This is the case if the surplus from trading in the digital sector Γ_d is sufficiently high compared to the physical sector surplus Γ_p such that the following inequality holds

$$\delta \left(N_{f,d}\right) - \delta' \left(N_{f,d}\right) N_{f,d} \geq \frac{1}{\alpha} \frac{\Gamma_p}{\Gamma_d}, \tag{34}$$

where $N_{f,d}$ is the solution to $\delta' \left(N_{f,d}\right) \Gamma_d = \frac{1}{\alpha\theta} \eta \left(N_{f,d}\right)$.

Similarly, an equilibrium where consumers only trade in the physical sector and thus $n_{c,p} = 1$ and $n_p = N_{f,p}$ exists if the surplus from trading in the physical sector Γ_p is sufficiently high compared to the digital sector surplus Γ_d such that the following inequality holds

$$\delta\left(N_{f,p}\right)-\delta'\left(N_{f,p}\right)N_{f,p}\geq\alpha\frac{\Gamma_d}{\Gamma_p},\tag{35}$$

and where $N_{f,p}$ is the solution to $\delta'\left(N_{f,p}\right)\Gamma_p=\left(\frac{1+i}{1+\alpha\theta i}\right)\eta\left(N_{f,p}\right)$.

In cases where firms enter both sectors, consumers must be indifferent between the two submarkets such that $V_{c,p}=V_{c,d}$ and thus

$$\alpha\left(\delta\left(n_d\right)-\delta'\left(n_d\right)n_d\right)\Gamma_d=\left(\delta\left(n_p\right)-\delta'\left(n_p\right)n_p\right)\Gamma_p.\tag{36}$$

Equation (36) highlights that in an equilibrium where both sectors are active, consumers trade off the gains from trade with the probability that they are matched with and are able to trade with a firm. In addition to the matching friction, absent CBDC, digital firms can only transact with consumers if they have access to the financial market; thus, in the case where the gains from trade are the same across sectors, consumers would require a higher probability of trade ($n_d>n_p$) in order to be indifferent between the two sectors.

Where firms enter both sectors, firm entry drives *ex ante* expected firm profits to zero. Given that firms must commit to paying a fixed cost to enter at the beginning of the period, all firms that enter a sector will also enter the submarket, thus $n_{f,j}=N_{f,j}\ \forall j$. Furthermore, since the gains from trade are positive, all consumers enter one of the two submarkets, and so $n_{c,d}+n_{c,p}=1$. Given this, $N_{f,d}$ and $N_{f,p}$ can be found as the solution to equations (28), (29), and (36).

In cases where $\alpha,\theta<1$, frictions in financial markets reduces the availability and increase the cost of digital payments. Money has a dual role to play in this case; first, it provides an alternative form of payment in cases where firms do not have access to card payments which facilitates a greater number of transactions, and second, it improves the bargaining position of firms in the financial market which allows more firms to enter the physical retail market.

In facilitating additional trades in the physical sector, money increases the probability that both firms and consumers are able to trade at a given submarket tightness relative to the digital sector. As a consequence, relatively more firms and relatively more consumers will enter the physical sector relative to the digital sector. From equation (35) it follows that if α is sufficiently small, only the physical sector will be active.

To highlight the role money plays in improving the bargaining position of firms in the physical sector, consider the special case where $\alpha=1$. This shuts down the first role that money plays in providing an alternative form of payment. Then in the symmetric case where $\kappa_p = \kappa_d$ it follows that $y_p > y_d$ for any $\theta < 1$ and $i < \infty$ and thus $\Gamma_p > \Gamma_d$. As firms in the physical sector face lower costs of financial intermediation, they pass this on to consumers in the form of higher gains from trade. For an interior solution, where both sectors are active, it follows from equation (36) that there must be a larger ratio of firms to consumers in the digital sector ($n_d > n_p$) to compensate for the lower gains from trade. As the matching function $\delta(\cdot)$ is concave, it then follows from equations (28) and (29) that a greater number of firms enter the physical sector than the digital sector, $N_{f,p} > N_{f,d}$, and consequently a larger number of consumers also enter the physical submarket compared to the digital submarket, $n_{c,p} > n_{c,d}$.

Turning now to the model's implications regarding the increase in the proportion of online sales identified in Figure 2, denote by Δ the share of total sales that take place in the digital sector, where

$$\Delta = \frac{\alpha n_{c,d}\delta(n_d)}{\alpha n_{c,d}\delta(n_d) + n_{c,p}\delta(n_p)}. \tag{37}$$

For simplicity and to generate analytical results, I focus here on the share of number of sales rather than the share of sales revenue. However, when I later calibrate the model to UK data, I use the model analogue to Figure 2, the share of sales revenue in the digital sector.

One possible explanation for the increased share of the digital retail sector is a relative increase in the productivity of the digital sector. Consider a decrease in κ_d, the marginal cost faced by firms in the digital sector. From equation (26) it follows that a decrease in κ_d results in high consumption in the digital sector, y_d, and thus higher gains from trade, Γ_d. This will result in some consumers switching from trading with physical firms to digital firms, leading to an increase in the share of sales made in the digital sector. This result is formalised in Proposition 1 below.

Proposition 1. *Consider a competitive equilibrium without CBDC such that $\gamma_d = 0$ and $\gamma_p = 1$. If the competitive equilibrium is such that $n_d > 0$ and $n_p > 0$, then decreasing the marginal cost of digital firms, κ_d: i) decreases $n_{c,p}$ and $N_{f,p}$; ii) increases $n_{c,d}$ and $N_{f,d}$; and iii) increases Δ.*

Proof. See the Appendix. □

Proposition 1 establishes that, absent CBDC, a decrease in the marginal cost of production for digital firms also affects the firm entry decision of firms. Driven by lower cost of production, the number of firms entering the digital sector increases, while higher competition and lower competitiveness leads to a fall in the number of firms entering the physical sector.

A second explanation for the increased share of the digital retail sector is a reduction in financial frictions. As discussed above, when $\alpha, \theta < 1$ frictions in financial markets result in digital firms being at a competitive disadvantage relative to physical firms. An increase in the availability of digital payments, α, or an increase in the firm's bargaining power, θ, will benefit digital firms relatively more than physical firms, leading to an increase in the share of digital sales increasing. Proposition 2 develops analytical results in the simplified case where $i \to 0$.

Proposition 2. *Consider a competitive equilibrium without CBDC such that $\gamma_d = 0$ and $\gamma_p = 1$. Assume $i \to 0$. If the competitive equilibrium is such that $n_d > 0$ and $n_p > 0$, then increasing either the firm's bargaining power, θ, or access to financial markets, α: i) decreases $n_{c,p}$ and $N_{f,p}$; ii) increases $n_{c,d}$ and $N_{f,d}$; and iii) increases Δ.*

Proof. See the Appendix. □

Proposition 2 establishes that, absent CBDC and in the case where $i \to 0$, an increase in α or θ reduces the competitive disadvantage facing digital firms, resulting in firm entry increasing in the digital sector and decreasing in the physical sector. Additional firm entry coupled with an increase in the gains from trade in the digital sector leads some consumers to switch from the physical sector to the digital sector as trading with digital firms becomes more appealing. The case where $i \to 0$ simplifies the proof as the gains in trade in the physical sector, Γ_p, are not affected by α or θ. In cases where $i > 0$, an increase in α or θ will increase both Γ_p and Γ_d. Intuitively, the relaxing financial frictions is likely to benefit the digital sector relatively more than the physical sector, as only the physical sector has the option to accept money as a means of payment. Thus, while no formal proof is offered, it is likely that this result will hold for reasonable values of $i > 0$.

3.2 Introduction of a CBDC

Consider now the case where the monetary authority decides to introduce a digital form of money, for example a CBDC, that can be used as a means of payment by digital firms. Thus the key change relative to the benchmark model is that equation (15) now becomes

$$\gamma_j = 1 \quad \forall i \in \{p, d\}\,. \tag{38}$$

The problem for digital firms becomes identical to the problem for physical firms, with the only difference between sectors occurring in cases where the sectors differ in terms of their marginal cost of production, and thus $\kappa_p \neq \kappa_d$.

Given that the introduction of a CBDC removes the disparity between the two sectors, we might expect to be able to reach the efficient equilibrium. From equation (27), it follows that if $\alpha\theta < 1$, the optimal level of consumption can be achieved for both sectors if $i \to 0$. Thus, the Friedman rule implements the optimal level of consumption. It follows that as $i \to 0$, the gains from trade are equal to the optimal surplus $\Gamma_j = S_j^*$. Furthermore, since equations (6) and (36) are identical, the market tightness n_j will also be optimal. However, firm entry will not be at the optimal level. This can be seen by comparing equation (5) with the analogue in the model with CBDC where $i \to 0$ implies that firm entry is determined by the following equation

$$\delta'(n_j)\,\Gamma_j = \eta(N_{f,j})\,. \tag{39}$$

Direct comparison of these equations highlights that entry will be higher than optimal in the model with CBDC and free entry of firms. This occurs because of the structure of entry costs, which is increasing in the number of entering firms. Formally, equation (39) lacks the term $N_{f,j}\eta'(N_{f,j})$ which captures the fact that in competitive equilibrium, firms do not internalise the effect their entry decision has on the entry cost of other firms. As a consequence, firm entry is too high relative to the optimum, and the Friedman rule does not yield the efficient allocation.

The efficient level of entry can be achieved through the use of a tax on firm entry. Specifically, consider a sector-specific entry tax τ_j that is levied on firms and redistributed to consumers through the lump sum transfer T. In the model with CBDC, the optimal level of entry can be achieved through setting the entry tax such that

$$(1+\tau_j)\,\eta(N_{f,j}) = \eta(N_{f,j}) + N_{f,j}\eta'(N_{f,j})\,. \tag{40}$$

The equilibrium with CBDC is optimal only if the government implements the Friedman rule, setting $i \to 0$ and taxing firm entry at a level such that firms internalise the entry externality. This result is formalised in Proposition 3 below.

Proposition 3. *A competitive equilibrium with CBDC such that $\gamma_d = 1$ and $\gamma_p = 1$ and financial frictions such that $\alpha\theta < 1$ is optimal iff i) $i \to 0$ and ii) there is a sector-specific tax on entry τ_j such that $\tau_j = N_{f,j}\eta'(N_{f,j})/\eta(N_{f,j})$.*

Proof. See the Appendix. □

A corollary of Proposition 3 is that, absent the optimal entry tax, the Friedman rule does not yield the optimal allocation. Instead, setting the optimal interest rate consists of a trade-off. While increasing the interest rate above zero reduces trade efficiency, it also raises the cost of entry for firms, mitigating the excessive entry.

The introduction of a CBDC eliminates the competitive advantage of the physical sector over the digital sector. Whether or not the introduction of a CBDC is welfare improving would depend in part on the severity of the entry externality. If the entry externality is sufficiently large, the introduction of a CBDC can, in increasing entry into the digital sector, reduce overall welfare. This scenario is likely only an academic curiosity that could be solved through the introduction of an entry tax.

I now consider whether the trend towards greater digital retail sales has affected the welfare gains of introducing a CBDC. Consider the case where the conditions set out in Proposition 3 hold and where introducing a CBDC yields the optimal allocation. In this case, whenever $\alpha\theta < 1$ the benchmark model will not yield the optimal allocation and thus introducing a CBDC will be strictly welfare increasing. Proposition 4 formalises this result in addition to how the welfare gain is affected by key model parameters.

Proposition 4. *In the case where $i \to 0$ and there is a sector-specific tax on firm entry $\tau_j = N_{f,j}\eta'\left(N_{f,j}\right)/\eta\left(N_{f,j}\right)$ then i) introducing a CBDC strictly increases welfare whenever $\alpha\theta < 1$; ii) the welfare gain of introducing a CBDC is decreasing in α and θ; and iii) if $\theta = 1$ and $\alpha < 1$ the welfare gain of introducing a CBDC is decreasing in κ_d.*

Proof. See the Appendix. □

Proposition 4 states that a reduction in financial frictions, either through an increase in α or θ, results in a decrease in the welfare gain of introducing a CBDC. Intuitively, the optimal welfare level remains unchanged after a change to α or θ, while, due to the reduction in financial frictions, there is an increase in the welfare of the equilibrium without CBDC.

Studying the welfare impacts of a decrease in marginal costs of digital firms, κ_d, is complicated by the fact that welfare increases for both the model with and without CBDC. Introducing the simplifying assumption $\theta = 1$ allows for analytic tractability. It follows from Proposition 4 that, conditional on $\theta = 1$, a fall in the marginal cost of digital firms, κ_d, increases the welfare gains from introducing a CBDC. As a decrease in κ_d leads to a shift from trades involving physical firms to digital firms, the welfare gain is greater if a CBDC has been introduced because digital firms face lower frictions than if a CBDC were absent.

The above discussion has an important implication regarding the possible benefits of introducing a CBDC in the context of the digitalisation trends identified by Figures 1 and 2. Should the increase in digital retail sales be driven by reductions in financial frictions, the benefits of introducing a CBDC would have decreased over time. However, if the increase in digital retail sales is driven by an increase in the productivity of digital firms, the welfare benefits of introducing a CBDC would increase as this trend continues. Which of these channels dominates is largely an empirical question.

4 Numerical analysis

The previous section set out two potential drivers of the digitisation of retail transactions; a reduction in financial frictions or an increase in the relative productivity of digital firms. However, these drivers have different implications for the welfare benefits of introducing a CBDC. This section provides quantitative results. First, the model is calibrated to UK data. The sensitivity of this baseline calibration is explored. Finally, the calibrated model is used to assess which of the drivers of digitisation were of the most quantitative importance, and thus how the welfare effects of the introduction of a CBDC have changed over time.

4.1 Baseline Calibration

The model is parameterised as follows. The utility function of consumers in the DM is

$$u\left(y_t\right) = \frac{1}{1-\sigma}\left[\left(y_t+\epsilon\right)^{1-\sigma} - \epsilon^{1-\sigma}\right], \tag{41}$$

where $\sigma > 0$ and the parameter ϵ is set to 0.0001. This ensures $u(0) = 0$ and $u(\cdot)$ is close to CRRA. This parameter has little impact quantitatively and is common in the money-search literature. The utility of agents in the CM is assumed to have the following functional form:

$$v_i\left(x\right) = B_i \ln\left(x\right), \tag{42}$$

which implies that $x^* = B_i$. In addition, it is assumed that $B = B_c = B_b$ while $\ln\left(B_f\right) = 1$ is such that the firm utility in the CM is normalised to zero. The matching function specified is analogous to the one used in Rocheteau and Wright (2009) and is

$$\delta\left(n_j\right) = \frac{n_j}{1+n_j}. \tag{43}$$

The firm entry cost is assumed to take the following quadratic form

$$\eta\left(N_{f,j}\right) = \frac{\eta_0}{2}\left(N_{f,j}\right)^2, \tag{44}$$

where $\eta_0 > 0$.

Table 1: Baseline Calibration

Parameters	Notation	Value	Calibration Target
Availability of card payments	α	0.467	Fraction of cash transactions 2010-2022
Digital firm cost	κ_d	0.601	Fraction of online transaction value 2010-2022
Firm bargaining power	θ	0.986	Card handling costs 2010-2022
Firm fixed cost	η_0	0.002	Retail Sector GFCF/GDP 2010-2022
Curvature of DM Consumption	σ	0.462	CIC/GDP demand function 1982-2023
Coefficient of CM Consumption	B	1.425	CIC/GDP demand function 1982-2023

The model is calibrated to UK annual data. The parameter that determines availability of card payments, α, is calibrated to the proportion of cash transactions relative to the total number of cash and card transactions. Data are obtained from UK Finance's UK Payment Markets Report, where the number of transactions made with cash averaged over 2010-2022 is around 48%. The marginal cost of physical firms, κ_p, is normalised to 1, while the marginal cost of digital firms, κ_d, is calibrated to the proportion of internet sales as a percentage of the total value of retail sales. Data are obtained from the ONS Retail Sales Index, where 16.4% of retail sales were made online on average between 2010 and 2022. The firm's bargaining power, θ, is calibrated to card handling costs as a fraction of the card transaction turnover. Data are obtained from the British Retail Consortium Payments Survey, where the average card handling costs during the 2010-2022 period stood at 0.41% of the turnover from card transactions. Finally, the fixed cost of the firm, η, is calibrated so that the ratio of fixed costs to total output in the model matches the average ratio of gross fixed capital in the retail sector to GDP between 2010 and 2022 (0.48%).

The parameters (σ, B) are calibrated analogously to those of Lucas (2000) and Lagos and Wright (2005) where the parameters are set to match the relationship between the nominal interest rate i and the demand for money $L \equiv \phi_t M_t / Y$ using a longer time series of data.

Output in terms of the numeraire good x in the decentralised market is

$$\bar{Y} = \alpha n_{c,d} \delta \left(n_d\right) \varphi_d y_d + n_{c,p} \delta \left(n_p\right) \varphi_p y_p, \tag{45}$$

and output in the centralised market is

$$\bar{X} = 2B + \left(N_{f,p} + N_{f,d}\right) B_f. \tag{46}$$

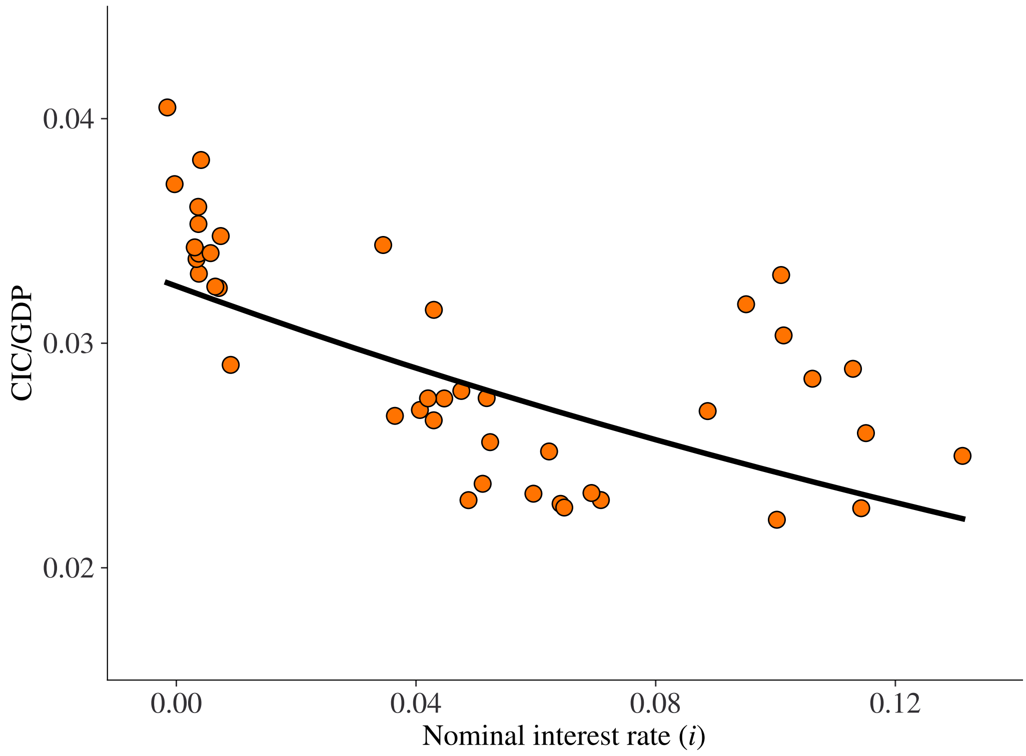

Figure 4: Model and Data (1982-2023)

Combining this with the equation for real balances (25) the model implied function for L is given by

$$L = \frac{\bar{M}}{\bar{Y} + \bar{X}} \tag{47}$$

where $\bar{M} = (1 - \alpha)\, n_{c,p} \delta\,(n_p)\, \varphi_{p,t} y_{p,t}$ is the value of money demanded in terms of the numeraire good x.

The series for i is the spot yield on UK one-year Gilts and PY is UK nominal GDP. I deviate from Lucas (2000) and Lagos and Wright (2005) letting M be the amount of currency in circulation (CIC) as opposed to M1. The reason behind this is that money in the model is best interpreted narrowly as cash, as opposed to M1. However, calibrating the model to M1 instead of CIC does not significantly alter the results.

Table 1 summarises the parameter values in the baseline calibration along with the corresponding calibration target. Figure 4 shows the model-predicted curve for CIC/GDP as a function of the nominal rate compared to the UK data between 1982 and 2023.

4.2 Effect of introducing a CBDC

I now consider the effect of the introduction of a CBDC into the model. The introduction of a CBDC allows digital firms to transact in the absence of access to the payments

	Data	Baseline Model	Model with CBDC	Optimal Allocation
Internet Sales	0.164	0.164	1.0	1.0
Cash Transactions	0.477	0.477	0.0	0.0
Card Handling Costs	0.0041	0.0041	0.004	0.0
Retail Sector GFCF / GDP	0.0048	0.0048	0.0105	0.0052
Welfare (optimal=100)		52	96	100

Table 2: Effect of CBDC introduction on Moments and Welfare

market. This has two effects: first, the number of possible transactions in the digital sector increases as transactions are no longer conditional on firms receiving access to the payments market, and second, digital firms have a better bargaining position with payment operators due to the existence of an outside option for transactions. The effect on welfare and key data moments of the introduction of a CBDC is presented in Table 2. The introduction of CBDC increases the proportion of sales in the digital sector and reduces the number of cash transactions that occur on average. The proportion of Internet sales following the introduction of CBDC is close to optimal. The model also predicts a significant increase in welfare of around 85%. Putting this welfare increase in context, the model abstracts away from many welfare costs associated with the introduction of CBDC in the literature, such as issues surrounding financial stability and bank disintermediation.

In the baseline calibration, the welfare gains are driven by the proportion of Internet sales being relatively low, while the number of cash transactions is relatively high. The model captures this through low availability of card payments ($\alpha = 0.467$) and with digital firms facing lower production costs compared to physical firms ($\kappa_d = 0.601$). Given that digital firms are more efficient than physical firms in this calibration, it would be optimal for digital firms to have a much larger share of sales revenue. This results in a large welfare loss that stems from digital firms without access to intermediaries not being able to transact with consumers. The introduction of a CBDC increases the ability of digital firms to transact with consumers, and thus the proportion of sales made by digital firms is close to the optimal level. The large welfare gains from introducing a CBDC fall, but do not disappear, if digital firms without access to financial intermediaries are allowed to exit the submarket after paying the entry cost but before they are matched with consumers. This alteration to the model timing does not qualitatively change the results of the model but does reduce the welfare gains from introducing a CBDC.

To better understand what generates the welfare gains from introducing a CBDC in the baseline calibration, Figure 5 compares the welfare of the model without CBDC (relative to optimal) with the welfare of the model with CBDC (relative to optimal), while altering the values of key parameters of the model. Other parameters are fixed at the calibration

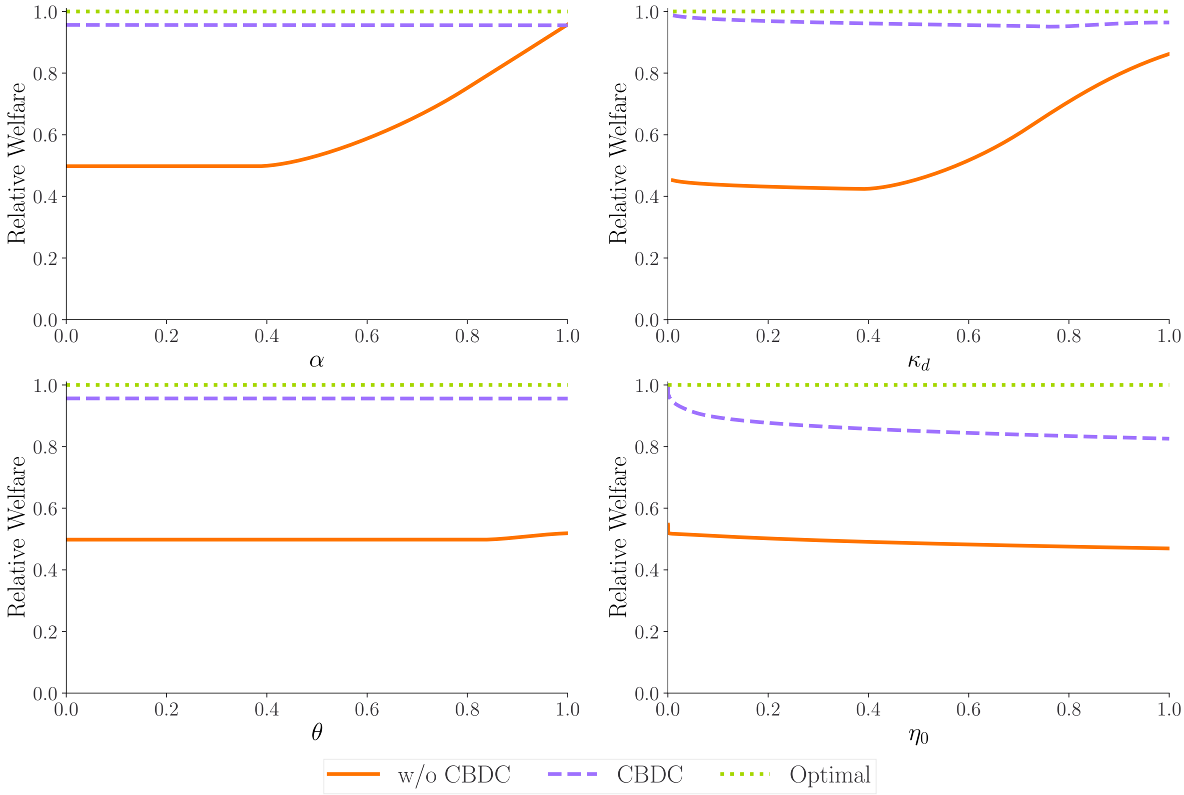

Figure 5: Sensitivity of welfare relative to optimal to Parameters

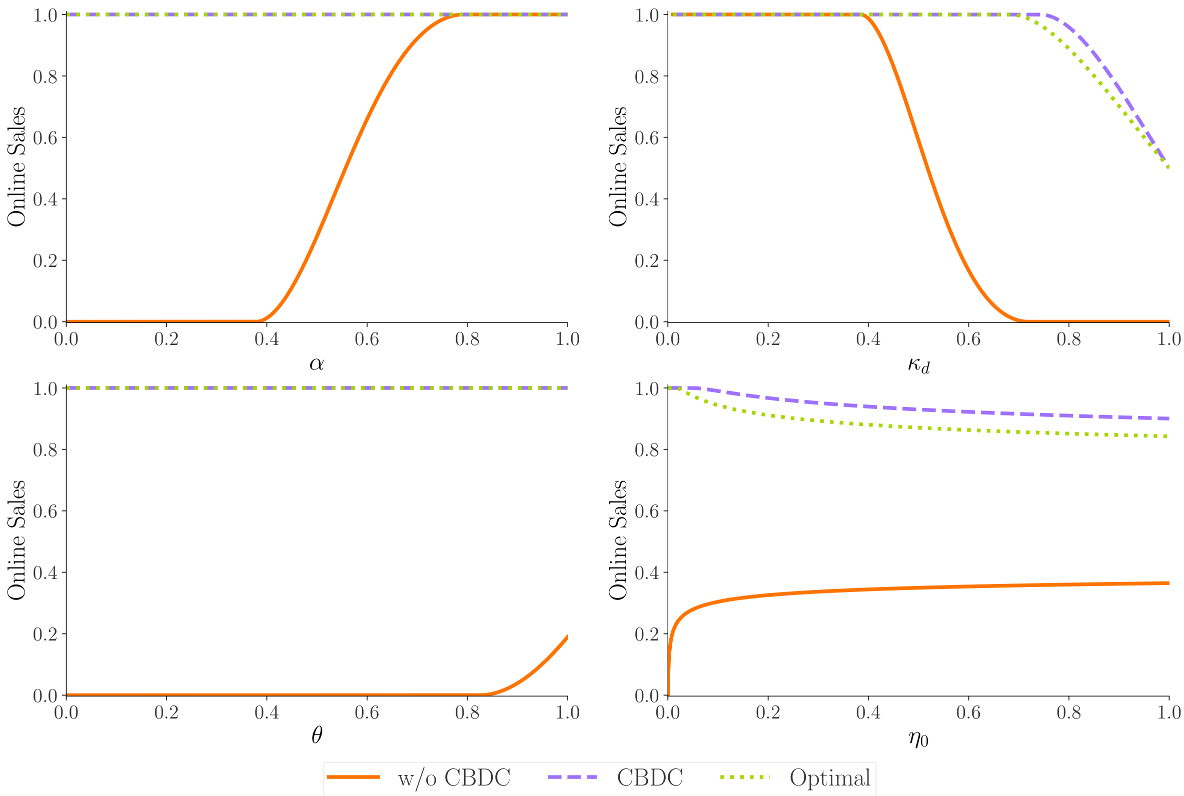

Figure 6: Sensitivity of share of digital sector to Parameters

values of the baseline model. Figure 5 highlights the sensitivity of welfare gains to the parameterisation. It should be noted that in the baseline calibration, interest rates are close to zero ($i = 0.006$) and θ is close to 1 and thus introducing a CBDC reduces most of the frictions in the model. The key difference in welfare between the CBDC model and the optimal allocation is the entry externality, which is comparatively small in welfare terms compared to the payment frictions. A key result is that welfare gains are largest when there is low availability of card payments (α is low) and when the relative productivity of the digital sector is high (κ_d is low). As $\alpha \to 1$ the model without CBDC approaches the optimal allocation and the welfare gains from introducing a CBDC disappear. In fact, as $\alpha \to 1$, there is a small welfare loss from the introduction of a CBDC. The reason for this is, due to the entry externality, there is too much entry in the model with CBDC. While the entry externality exists in the benchmark model, the over-entry is mitigated by the lack of firm bargaining power in the digital sector.

4.3 Capturing the rise of the digital economy

In the baseline calibration, the model was matched to the 2010-2022 average data, where the share of Internet sales was relatively low and the share of cash transactions relatively high compared to the latest data. As illustrated in Figures 1 and 2, this period featured a rapid increase in both the proportion of Internet retail sales and a decrease in the share of cash transactions in the UK.

Figure 6 shows how the fraction of sales made in the digital sector varies as key parameters of the model are altered, holding other parameters fixed at the calibration values of the baseline model. The model suggests that several parameters could be driving the increasing trend in online retail observed in the data. First, a decrease in κ_d relative to κ_p would result in digital firms facing lower marginal production costs and hence are more productive than physical firms. This could occur due to technological advances being made in the economy that benefit digital firms to a greater extent than physical stores. This increase in relative productivity would lead to a higher proportion of consumers entering the digital submarket and would lead to digital firms receiving a larger share of total trades. Second, an increase in availability of card payments, α, would result in digital firms having a higher probability to transact with consumers, and as a consequence consumers would switch from physical stores to digital stores. Although an increase in both the entry cost, η_0, and the firm's bargaining share, θ, could also explain the online retail trend, the model appears less sensitive to these parameters at the benchmark calibration, and thus it is unlikely that these would be the key drivers of the online retail trend.

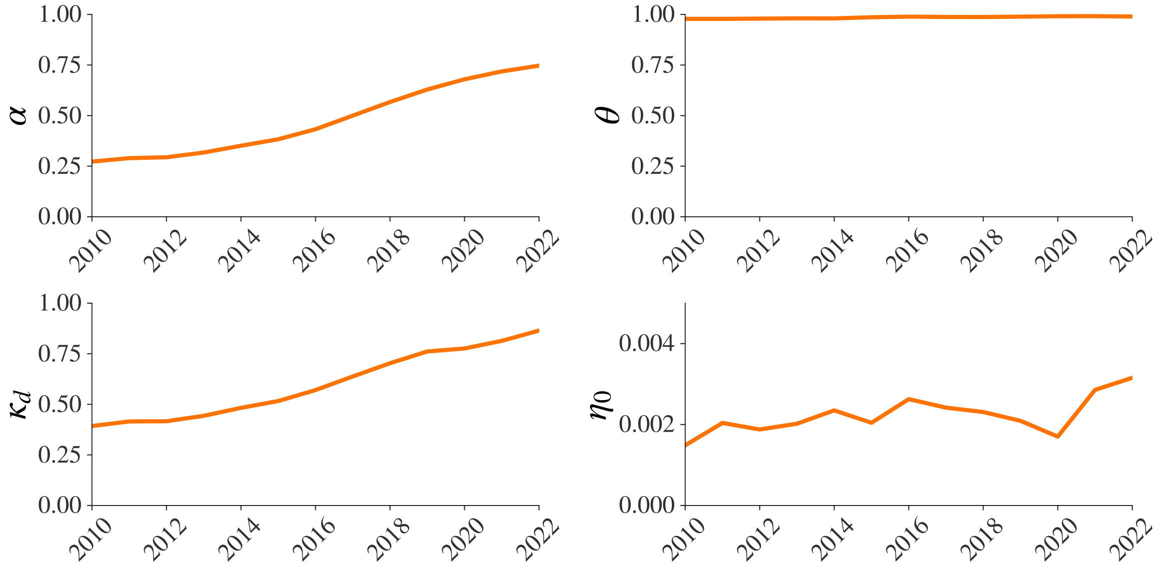

Figure 7: Parameter Calibration (2010-2022)

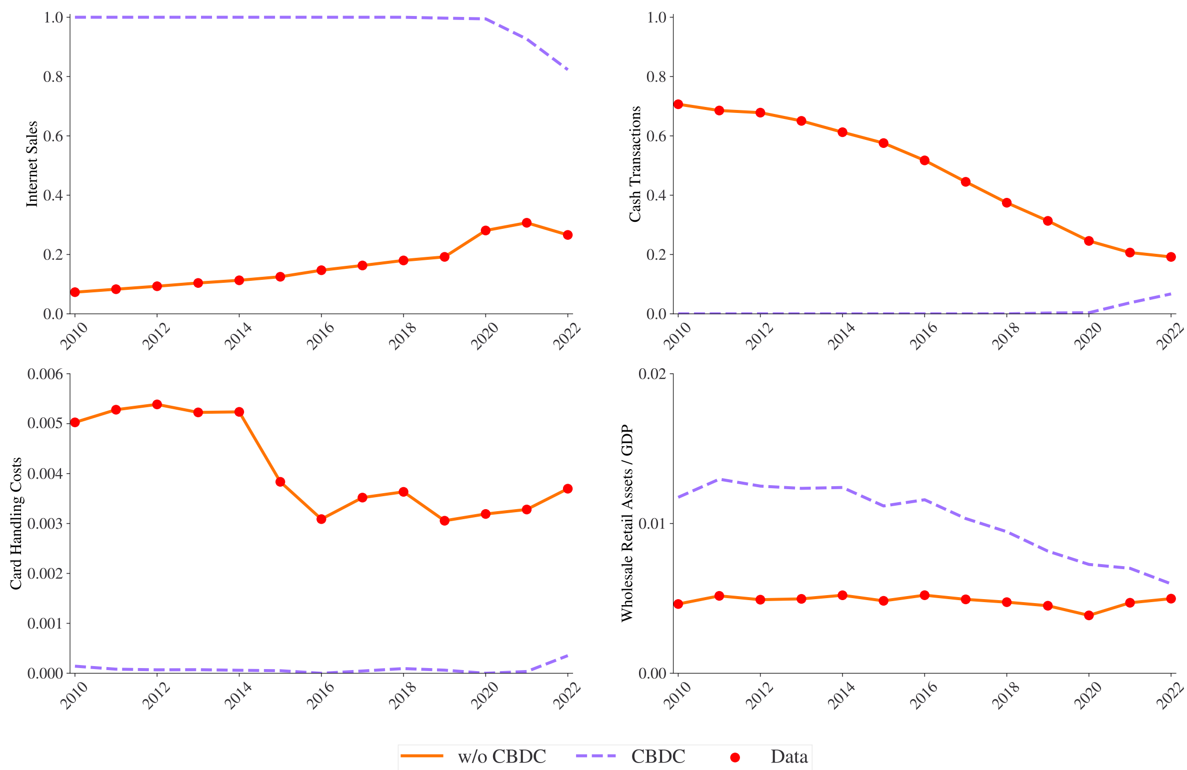

Figure 8: Calibrated Moments and Counterfactual Moments (2010-2022)

Focussing on an increase in the availability of card payments α and a decrease in marginal cost in the digital sector κ_d, it is worth noting that these two drivers toward online retail have very different implications for the welfare gains of introducing a CBDC. A decrease in κ_d would increase the welfare gains from the introduction of a CBDC while an increase in α would decrease the welfare gains from the introduction of a CBDC. To address which of these channels dominates, I now extend the baseline calibration by allowing the parameters $(\alpha_t, \kappa_{d,t}, \theta_t, \eta_{0,t})$ to vary over the years 2010-2022 while keeping the remaining parameters fixed at their values in the baseline calibration. The parameters α_t, $\kappa_{d,t}$, $\eta_{0,t}$ and θ_t are calibrated to the time series of the proportion of cash transactions, the proportion of Internet sales, retail sector GFCF / GDP, and card handling costs, respectively.

Figure 7 shows the series of calibrated parameters. The model suggests that the increase in the share of Internet sales and the decrease in cash use are driven by an increase in α, greater availability of card payments, as opposed to lower marginal costs of digital firms, κ_d. In fact, the model suggests that the marginal costs of digital firms increased over the period 2010-2022.

Figure 8 shows the calibrated moments for the benchmark model and the model with CBDC. The benchmark model matches the increase in Internet sales and the decline in cash transactions found in the data. Cash transactions in the CBDC model capture payments in the physical sector made using money which, given that the model does not distinguish between the two, could consist of both cash payments and CBDC payments. As the model suggests κ_d increases during the 2010-2022 period, the gap between the benchmark model and the model with CBDC narrows for both Internet sales and cash transactions.

The implications of this calibration are that the welfare gains of introducing a CBDC fall over the period 2010-2022. Figure 9 shows that welfare in the benchmark model relative to optimal increases over time, while welfare in the model with CBDC remains close to optimal. The gain in relative welfare in the benchmark model is driven by the increase in both α and κ_d over the time period. This suggests that the decline in cash and increase in online sales observed in the data is driven predominantly by improvements in payment efficiency rather than productivity gains in digital retail. If anything, the productivity gap between online and offline retail appears to be narrowing over time. The relative welfare of the model with CBDC remains close to optimal because the nominal interest rates are close to zero for most of the period, while θ also remains close to 1.

As discussed previously, since the model abstracts from possible negative implications of introducing a CBDC, the level of welfare gains of introducing a CBDC to the model should only be interpreted as measuring the possible benefit of offering an alternative

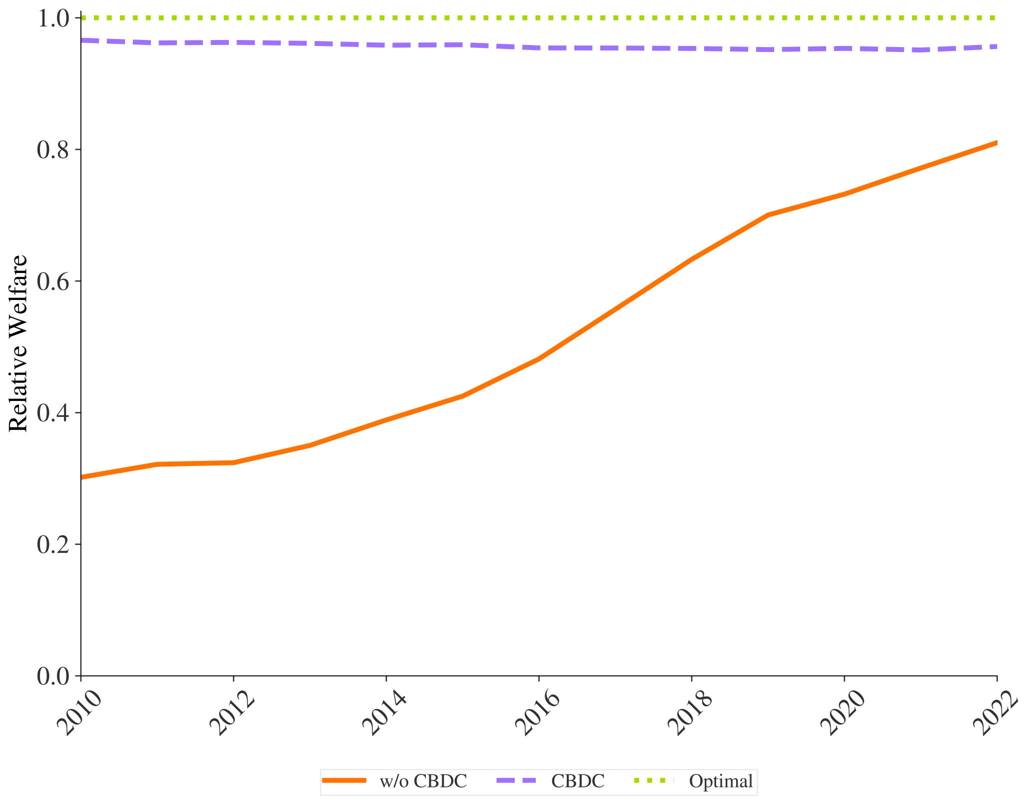

Figure 9: Welfare relative to Optimal (2010-2022)

means of payment for digital transactions. Given the benefit of introducing a CBDC benefits firms in the digital sector, one might think that the welfare gains would be greater when there are a higher number of Internet transactions and fewer transactions made using cash. The results presented here suggest that this may not necessarily be the case. Instead, it is important to gain an understanding of what is driving the shift toward Internet sales and away from cash transactions.

5 Conclusion

Current proposals for the introduction of a CBDC, such as those made by the Bank of England and the ECB, have coalesced around an unremunerated CBDC. An obvious question then is, if it offers the same rate of return, how would a CBDC be distinct from existing forms of money? This paper proposes a simple mechanism through which an unremunerated CBDC may improve welfare by lowering the market power financial intermediaries have in negotiating fees with digital firms that cannot use cash as an alternative means of payment.

Although the introduction of a CBDC has the potential to improve welfare, the model suggests that the size of any welfare increase depends on the degree to which the inability to trade with physical money deters the entry of online retailers. Given the trend for an increasing proportion of online retail transactions and the increasing use of digital payment methods, it is important to understand the underlying drivers of these changes. When calibrated to the UK data, the model suggests that while there are still significant welfare gains from the introduction of a CBDC, the benefits have actually fallen over the period 2010 to 2022 as the proportion of online transactions has increased and cash usage has fallen.

Finally, while the model presented in this document abstracts from many of the design choices central banks are currently facing, the basic mechanism of the model relies on a CBDC as offering a credible alternative to existing digital payment methods. Thus, should a CBDC reuse existing payment infrastructure without providing additional competition, the welfare benefits set out in this paper would be overstated.

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Appendix

Comparative Statics

A competitive equilibrium such that $n_d > 0$ and $n_p > 0$ satisfies equations (28), (29) and (36). Using the fact that by definition $n_{f,j} = N_{f,j} = n_{c,j} n_j$ and that in any equilibrium all consumers attempt to trade, $n_{c,d} + n_{c,p} = 1$, the equilibrium can be written as the following system of three equations and three endogenous variables $\mathbf{n} = (n_p, n_d, n_{c,p})$:

$$\mathbf{F} \equiv \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} (\alpha + \gamma_d (1-\alpha)) (\delta(n_d) - n_d \delta'(n_d)) \Gamma_d - (\delta(n_p) - n_p \delta'(n_p)) \Gamma_p \\ (\alpha + \gamma_d (1-\alpha)) \delta'(n_d) \Gamma_d - \left((1-\gamma_d) \frac{1}{\theta} + \gamma_d \left(\frac{1+i}{1+\alpha\theta i}\right)\right) \eta ((1-n_{c,p}) n_d) \\ \delta'(n_p) \Gamma_p - \left(\frac{1+i}{1+\alpha\theta i}\right) \eta (n_{c,p} n_p) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{A.48}$$

where $\Gamma_j \equiv u(y_j) - y_j u'(y_j)$ and

$$y_j = u'^{-1} \left(\left((1-\gamma_d) \frac{1}{\theta} + \gamma_d \left(\frac{1+i}{1+\alpha\theta i} \right) \right) \kappa_j \right). \tag{A.49}$$

Here we have already imposed $\gamma_p = 1$ as the physical sector is assumed to always be able to accept cash.

The Jacobian of $\mathbf{F}$ can be written as

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial n_d} & \frac{\partial f_1}{\partial n_p} & 0 \\ \frac{\partial f_2}{\partial n_d} & 0 & \frac{\partial f_2}{\partial n_{c,p}} \\ 0 & \frac{\partial f_3}{\partial n_p} & \frac{\partial f_3}{\partial n_{c,p}} \end{bmatrix} \tag{A.50}$$

where it follows from the derivatives of $\mathbf{F}$ that $\det(J) > 0$.

Next, denote by $J_{n_i,z}$ matrix formed by replacing the i-th column of J by the column vector $\left[\frac{\partial f_1}{\partial z} \frac{\partial f_2}{\partial z} \frac{\partial f_3}{\partial z}\right]'$ where z is some exogenous variable. The derivative of n_i with respect to z is then

$$\frac{\partial n_i}{\partial z} = -\frac{\det(J_{n_i,z})}{\det(J)} \tag{A.51}$$

Proof of Proposition 1

In a competitive equilbirium without CBDC, $\gamma_d = 0$ and thus for some exogenous variable z we have the following equations for $J_{n_i,z}$

$$\begin{aligned}\det\left(J_{n_d,z}\right) &= -\frac{\partial f_1}{\partial z}\left(\delta''\left(n_p\right)\Gamma_p - \left(\frac{1+i}{1+\alpha\theta i}\right)n_{c,p}\eta'_p\right)\frac{1}{\theta}n_d\eta'_d \\ &\quad + \delta''\left(n_p\right)n_p\Gamma_p\left(\frac{\partial f_2}{\partial z}\left(\frac{1+i}{1+\alpha\theta i}\right)n_p\eta'_p + \frac{\partial f_3}{\partial z}\frac{1}{\theta}n_d\eta'_d\right)\end{aligned} \tag{A.52}$$

$$\begin{aligned}\det\left(J_{n_p,z}\right) &= \alpha\delta''\left(n_d\right)n_d\Gamma_d\left(\frac{\partial f_2}{\partial z}\left(\frac{1+i}{1+\alpha\theta i}\right)n_p\eta'_p + \frac{\partial f_3}{\partial z}\frac{1}{\theta}n_d\eta'_d\right) \\ &\quad + \frac{\partial f_1}{\partial z}\left(\alpha\delta''\left(n_d\right)\Gamma_d - \frac{1}{\theta}\left(1-n_{c,p}\right)\eta'_d\right)\left(\frac{1+i}{1+\alpha\theta i}\right)n_p\eta'_p\end{aligned} \tag{A.53}$$

$$\begin{aligned}\det\left(J_{n_{c,p},z}\right) &= \frac{\partial f_2}{\partial z}\left(\delta''\left(n_p\right)\Gamma_p - \left(\frac{1+i}{1+\alpha\theta i}\right)n_{c,p}\eta'_p\right)\alpha\delta''\left(n_d\right)n_d\Gamma_d \\ &\quad - \frac{\partial f_3}{\partial z}\delta''\left(n_p\right)n_p\Gamma_p\left(\alpha\delta''\left(n_d\right)\Gamma_d - \frac{1}{\theta}\left(1-n_{c,p}\right)\eta'_d\right) \\ &\quad + \frac{\partial f_1}{\partial z}\left(\alpha\delta''\left(n_d\right)\Gamma_d - \frac{1}{\theta}\left(1-n_{c,p}\right)\eta'_d\right)\left(\delta''\left(n_p\right)\Gamma_p - \left(\frac{1+i}{1+\alpha\theta i}\right)n_{c,p}\eta'_p\right)\end{aligned} \tag{A.54}$$

where $\eta'_j = \eta'\left(N_{f,j}\right)$.

Differentiating F with respect to κ_d we note that $\frac{\partial f_1}{\partial \kappa_d} < 0$, $\frac{\partial f_2}{\partial \kappa_d} < 0$, $\frac{\partial f_3}{\partial \kappa_d} = 0$. It follows from equation (A.51) that $\frac{\partial n_{c,p}}{\partial \kappa_d} > 0$. Note also that as $n_{c,d} = 1 - n_{c,p}$ it also follows that $\frac{\partial n_{c,d}}{\partial \kappa_d} < 0$.

Consider the equation for proportion of online trades

$$\Delta = \alpha\frac{(1-\zeta)\,\delta\left(n_d\right)}{\alpha\left(1-\zeta\right)\delta\left(n_d\right) + \zeta\delta\left(n_p\right)} \tag{A.55}$$

Note that we can take derivatives of this with respect to a generic exogenous variable z.

$$\frac{d\Delta}{dz} = \frac{\partial\Delta}{\partial n_d}\frac{\partial n_d}{\partial z} + \frac{\partial\Delta}{\partial n_p}\frac{\partial n_p}{\partial z} + \frac{\partial\Delta}{\partial\zeta}\frac{\partial\zeta}{\partial z} + \frac{\partial\Delta}{\partial z} \tag{A.56}$$

or alternatively

$$\frac{d\Delta}{dz} = -\frac{1}{\det\left(J\right)}\left(\frac{\partial\Delta}{\partial n_d}\det\left(J_{n_d,z}\right) + \frac{\partial\Delta}{\partial n_p}\det\left(J_{n_p,z}\right) + \frac{\partial\Delta}{\partial\zeta}\det\left(J_{n_{c,p},z}\right)\right) + \frac{\partial\Delta}{\partial z} \tag{A.57}$$

Note that by collecting terms we can rewrite this in the form

$$\frac{d\Delta}{dz} = \frac{\partial f_1}{\partial z}\frac{B_1}{B_0} + \frac{\partial f_2}{\partial z}\frac{B_2}{B_0} - \frac{\partial f_3}{\partial z}\frac{B_3}{B_0} + \frac{\partial\Delta}{\partial z} \tag{A.58}$$

where

$$B_0 = (\alpha (1 - n_{c,p}) \delta (n_d) + n_{c,p}\delta (n_p))^2 \det (J) > 0 \tag{A.59}$$

$$\begin{aligned} B_1 = &- (1 - n_{c,p}) n_{c,p} \frac{\partial f_2}{\partial n_{c,p}} \frac{\partial f_3}{\partial n_p} \delta (n_p) \left(\frac{\delta (n_d)}{n_d} - \delta' (n_d) \right) \\ &- n_{c,p} \frac{\partial f_1}{\partial n_d} \frac{\partial f_3}{\partial n_p} \delta (n_p) \frac{\delta (n_d)}{n_d} \\ &+ (1 - n_{c,p}) n_{c,p} \frac{\partial f_2}{\partial n_d} \frac{\partial f_3}{\partial n_{c,p}} \delta (n_d) \left(\frac{\delta (n_p)}{n_p} - \delta' (n_p) \right) \\ &+ (1 - n_{c,p}) \frac{\partial f_2}{\partial n_d} \frac{\partial f_1}{\partial n_p} \delta (n_d) \frac{\delta (n_p)}{n_p} \end{aligned} \tag{A.60}$$

$$\begin{aligned} B_2 = &- n_{c,p} \frac{\partial f_1}{\partial n_d} \frac{\partial f_3}{\partial n_{c,p}} \left(\frac{\delta (n_p)}{n_p} - (1 - n_{c,p}) \delta' (n_p) \right) \\ &- \frac{\partial f_1}{\partial n_d} \frac{\partial f_1}{\partial n_p} \delta (n_d) \delta (n_p) \\ &+ (1 - n_{c,p}) n_{c,p} \frac{\partial f_1}{\partial n_p} \frac{\partial f_3}{\partial n_{c,p}} \delta' (n_d) \delta (n_p) \end{aligned} \tag{A.61}$$

$$\begin{aligned} B_3 = &- (1 - n_{c,p}) \frac{\partial f_1}{\partial n_p} \frac{\partial f_2}{\partial n_{c,p}} \left(\frac{\delta (n_d)}{n_d} - n_{c,p}\delta' (n_d) \right) \\ &- \frac{\partial f_1}{\partial n_d} \frac{\partial f_1}{\partial n_p} \delta (n_d) \delta (n_p) \\ &+ (1 - n_{c,p}) n_{c,p} \frac{\partial f_1}{\partial n_d} \frac{\partial f_2}{\partial n_{c,p}} \delta (n_d) \delta' (n_p) \end{aligned} \tag{A.62}$$

and $B_0, B_1, B_2, B_3 > 0$.

First, note that $\frac{\partial \Delta}{\partial \kappa_d} = 0$ and thus, using the fact that $\frac{\partial f_1}{\partial \kappa_d} < 0$, $\frac{\partial f_2}{\partial \kappa_d} < 0$, $\frac{\partial f_3}{\partial \kappa_d} = 0$ it follows from equation (A.58) that $\frac{d\Delta}{d\kappa_d} < 0$.

Turning to the entry of digital firms and noting that, $N_{f,d} = (1 - n_{c,p}) n_d$, the derivative of $N_{f,d}$ with respect to an exogenous variable z is

$$\frac{\partial N_{f,d}}{\partial z} = (1 - n_{c,p}) \frac{\partial n_d}{\partial z} - \frac{\partial n_{c,p}}{\partial z} n_d \tag{A.63}$$

and thus

$$\frac{\partial N_{f,d}}{\partial z} = \frac{1}{\det (J)} \left(n_d \det \left(J_{n_{c,p},z} \right) - (1 - n_{c,p}) \det (J_{n_d,z}) \right) \tag{A.64}$$

which, following some rearranging, can be expressed as

$$\frac{\partial N_{f,d}}{\partial z} = \frac{1}{\det (J)} \left(\frac{\partial f_1}{\partial z} C_{1,d} + \frac{\partial f_2}{\partial z} C_{2,d} - \frac{\partial f_3}{\partial z} C_{3,d} \right) \tag{A.65}$$

where

$$C_{1,d} = n_d \left(\alpha\delta'' (n_d) \Gamma_d - \frac{1}{\theta} (1 - n_{c,p}) \eta_d' \right) \delta'' (n_p) \Gamma_p \tag{A.66}$$

$$C_{2,d} = n_d \left(\delta'' \left(n_p \right) \Gamma_p - \left(\frac{1+i}{1+\alpha\theta i} \right) n_{c,p} \eta_p' \right) \alpha \delta'' \left(n_d \right) n_d \Gamma_d \\ - \left(1 - n_{c,p} \right) \delta'' \left(n_p \right) n_p \Gamma_p \left(\frac{1+i}{1+\alpha\theta i} \right) n_p \eta_p' \tag{A.67}$$

$$C_{3,d} = n_d \delta'' \left(n_p \right) n_p \Gamma_p \alpha \delta'' \left(n_d \right) \Gamma_d \tag{A.68}$$

and $C_{1,d}, C_{2,d}, C_{3,d} > 0$. Again, using the fact that $\frac{\partial f_1}{\partial \kappa_d} < 0$, $\frac{\partial f_2}{\partial \kappa_d} < 0$, $\frac{\partial f_3}{\partial \kappa_d} = 0$ it follows from equation (A.65) that $\frac{\partial N_{f,d}}{\partial \kappa_d} < 0$.

Finally, turning to the entry of physical firms and noting that $N_{f,p} = n_{c,p} n_p$, the derivative with respect to an exogenous variable z is

$$\frac{\partial N_{f,p}}{\partial z} = \frac{\partial n_{c,p}}{\partial z} n_p + n_{c,p} \frac{\partial n_p}{\partial z} \tag{A.69}$$

and thus

$$\frac{\partial N_{f,p}}{\partial z} = -\frac{1}{\det \left(J \right)} \left(n_p \det \left(J_{n_{c,p},z} \right) + n_{c,p} \det \left(J_{n_d,z} \right) \right) \tag{A.70}$$

which, following some rearranging, can be expressed as

$$\frac{\partial N_{f,p}}{\partial z} = -\frac{\partial f_1}{\partial z} C_{1,p} - \frac{\partial f_2}{\partial z} C_{2,p} + \frac{\partial f_3}{\partial z} C_{3,p} \tag{A.71}$$

where

$$C_{1,p} = n_p \left(\alpha \delta'' \left(n_d \right) \Gamma_d - \frac{1}{\theta} \left(1 - n_{c,p} \right) \eta_d' \right) \delta'' \left(n_p \right) \Gamma_p \tag{A.72}$$

$$C_{2,p} = n_p \delta'' \left(n_p \right) \Gamma_p \alpha \delta'' \left(n_d \right) n_d \Gamma_d \tag{A.73}$$

$$C_{3,p} = n_p \delta'' \left(n_p \right) n_p \Gamma_p \left(\alpha \delta'' \left(n_d \right) \Gamma_d - \frac{1}{\theta} \left(1 - n_{c,p} \right) \eta_d' \right) \\ - n_{c,p} \alpha \delta'' \left(n_d \right) n_d \Gamma_d \frac{1}{\theta} n_d \eta_d' \tag{A.74}$$

and $C_{1,p}, C_{2,p}, C_{3,p} > 0$. Again, using the fact that $\frac{\partial f_1}{\partial \kappa_d} < 0$, $\frac{\partial f_2}{\partial \kappa_d} < 0$, $\frac{\partial f_3}{\partial \kappa_d} = 0$ it follows from equation (A.71) that $\frac{\partial N_{f,p}}{\partial \kappa_d} > 0$. This completes the proof of proposition 1.

Proof of Proposition 2

First, consider the effect of a change in α. Using the fact that in equilibrium, $f_1 = 0$, the partial derivative of f_1 with respect to α can be written

$$\frac{\partial f_1}{\partial \alpha} = \frac{1}{\alpha} \left(\delta \left(n_p \right) - n_p \delta' \left(n_p \right) \right) \left(u \left(y_p \right) - \left(\frac{1+i+\alpha\theta i}{1+\alpha\theta i} \right) u' \left(y_p \right) y_p \right) \tag{A.75}$$

Similarly,

$$\frac{\partial f_2}{\partial \alpha} = \delta'(n_d)\,\Gamma_d > 0, \tag{A.76}$$

and using the fact that $f_3 = 0$

$$\frac{\partial f_3}{\partial \alpha} = \left(\frac{\theta i}{1+\alpha\theta i}\right)\delta'(n_p)\,\Gamma_p \geq 0. \tag{A.77}$$

Now taking the limit as $i \to 0$ it follows that $\frac{\partial f_1}{\partial \alpha} > 0$ and $\frac{\partial f_3}{\partial \alpha} = 0$. From equation (A.51) it follows that $\lim_{i\to 0}\left\{\frac{\partial n_{c,p}}{\partial \alpha}\right\} < 0$ and also that $\lim_{i\to 0}\left\{\frac{\partial n_{c,d}}{\partial \alpha}\right\} > 0$. It follows from equation (A.65) and equation (A.71) that $\lim_{i\to 0}\left\{\frac{\partial N_{f,d}}{\partial \alpha}\right\} > 0$ and $\lim_{i\to 0}\left\{\frac{\partial N_{f,p}}{\partial \alpha}\right\} < 0$. Note also that the partial derivative of Δ with respect to α is positive and thus it follows from equation (A.58) it follows that $\lim_{i\to 0}\left\{\frac{d\Delta}{d\alpha}\right\} > 0$.

Similarly for θ, the partial derivative of f_1 with respect to θ is

$$\frac{\partial f_1}{\partial \theta} = \alpha\left(\delta(n_d) - n_d\delta'(n_d)\right)\frac{1}{\theta}u'(y_d)\,y_d - \left(\delta(n_p) - n_p\delta'(n_p)\right)\left(\frac{\alpha i}{1+\alpha\theta i}\right)u'(y_p)\,y_p. \tag{A.78}$$

Similarly using the fact that $f_2 = 0$ in equilibrium, derivative with respect to

$$\frac{\partial f_2}{\partial \theta} = \alpha\frac{1}{\theta}\delta'(n_d)\left(u(y_d) - \left(\frac{1-\theta}{\theta}\right)u'(y_d)\,y_d\right) > 0, \tag{A.79}$$

and

$$\frac{\partial f_3}{\partial \theta} = \delta'(n_p)\left(\frac{\alpha i}{1+\alpha\theta i}\right)\left(u(y_p) - \left(\frac{(1-\alpha\theta)\,i}{1+\alpha\theta i}\right)u'(y_p)\,y_p\right) \geq 0. \tag{A.80}$$

Now taking the limit as $i \to 0$ it follows that $\frac{\partial f_1}{\partial \theta} = 0$ and $\frac{\partial f_3}{\partial \alpha} = 0$. From equation (A.51) it follows that $\lim_{i\to 0}\left\{\frac{\partial n_{c,p}}{\partial \theta}\right\} < 0$ and also that $\lim_{i\to 0}\left\{\frac{\partial n_{c,d}}{\partial \theta}\right\} > 0$. It follows from equation (A.65) and equation (A.71) that $\lim_{i\to 0}\left\{\frac{\partial N_{f,d}}{\partial \theta}\right\} > 0$ and $\lim_{i\to 0}\left\{\frac{\partial N_{f,p}}{\partial \theta}\right\} < 0$. Note also that the partial derivative of Δ with respect to α is positive and thus it follows from equation (A.58) it follows that $\lim_{i\to 0}\left\{\frac{d\Delta}{d\theta}\right\} > 0$. This completes the proof of propostion 2.

Proof of Proposition 3

Denote by $\bar{\mathcal{W}}$ the welfare of an equilibrium with CBDC such that $\gamma_d = 1$. It will be useful to rewrite this in terms of $N_{f,d}$, n_p, and $n_{c,p}$, in which case this is

$$\begin{aligned}\bar{\mathcal{W}} =& n_{c,p}\delta\left(n_p\right)\left(u\left(y_p\right) - \left(1 + (1-\alpha)\left(\frac{i}{1+i}\right)\right)\kappa_p \bar{y}_p\right) \\ &- n_{c,p}n_p\left(1 + (1-\alpha)\left(\frac{i}{1+\alpha\theta i}\right)\right)\eta\left(n_{c,p}n_p\right) \\ &+ \left(1-n_{c,p}\right)\delta\left(\frac{N_{f,d}}{1-n_{c,p}}\right)\left(u\left(y_d\right) - \left(1 + (1-\alpha)\left(\frac{i}{1+i}\right)\right)\kappa_d \bar{y}_d\right) \\ &- \left(1-n_{c,p}\right)n_d\left(1 + (1-\alpha)\left(\frac{i}{1+\alpha\theta i}\right)\right)\eta\left(N_{f,d}\right),\end{aligned} \tag{A.81}$$

where

$$\bar{y}_j = u'^{-1}\left(\frac{1+i}{1+\alpha\theta i}\right)\kappa_j. \tag{A.82}$$

As shown in section 2.2, in the case where $\alpha\theta < 1$ the optimal level of consumption in both sectors is achieved through setting $i \to 0$. Similarly, the optimal level of entry can only be achieved by setting a sector specific tax on entry such that

$$\tau_j = \frac{N_{f,j}\eta'\left(N_{f,j}\right)}{\eta\left(N_{f,j}\right)}. \tag{A.83}$$

Proof of Proposition 4

Consider now the welfare of an equilibrium without CBDC such that $\gamma_d = 0$ and denote this $\mathcal{W}$. This can be rewritten in terms of $N_{f,d}$, n_p, and $n_{c,p}$ as

$$\begin{aligned}\mathcal{W} =& n_{c,p}\delta\left(n_p\right)\left(u\left(y_p\right) - \left(1 + (1-\alpha)\left(\frac{i}{1+i}\right)\right)\kappa_p y_p\right) \\ &- n_{c,p}n_p\left(1 + (1-\alpha)\left(\frac{i}{1+\alpha\theta i}\right)\right)\eta\left(n_{c,p}n_p\right) \\ &+ \alpha\left(1-n_{c,p}\right)\delta\left(\frac{N_{f,d}}{1-n_{c,p}}\right)\left(u\left(y_d\right) - \kappa_d y_d\right) \\ &- N_{f,d}\eta\left(N_{f,d}\right)\end{aligned} \tag{A.84}$$

Now consider the derivatives of this with respect to α, κ_d, and θ.

$$\frac{\partial \mathcal{W}}{\partial \alpha} = \left(1-n_{c,p}\right)\delta\left(n_d\right)\left(u'\left(y_d\right) - \kappa_d y_d\right) + \frac{\partial \mathcal{W}}{\partial n_{c,p}}\frac{\partial n_{c,p}}{\partial \alpha} + \frac{\partial \mathcal{W}}{\partial N_{f,d}}\frac{\partial N_{f,d}}{\partial \alpha} > 0 \tag{A.85}$$

$$\frac{\partial \mathcal{W}}{\partial \theta} = \frac{\partial \mathcal{W}}{\partial n_{c,p}}\frac{\partial n_{c,p}}{\partial \theta} + \frac{\partial \mathcal{W}}{\partial N_{f,d}}\frac{\partial N_{f,d}}{\partial \theta} + \frac{\partial \mathcal{W}}{\partial y_d}\frac{\partial y_d}{\partial \theta} \geq 0 \tag{A.86}$$

$$\frac{\partial \mathcal{W}}{\partial \kappa_d} = -\alpha\left(1-n_{c,p}\right)\delta\left(n_d\right)y_d + \frac{\partial \mathcal{W}}{\partial n_{c,p}}\frac{\partial n_{c,p}}{\partial \kappa_d} + \frac{\partial \mathcal{W}}{\partial N_{f,d}}\frac{\partial N_{f,d}}{\partial \kappa_d} + \frac{\partial \mathcal{W}}{\partial y_d}\frac{\partial y_d}{\partial \kappa_d} < 0 \tag{A.87}$$

where we note that we have the following derivatives in the limit as $i \to 0$

$$\lim_{i\to 0}\left\{\frac{\partial \mathcal{W}}{\partial n_{c,p}}\right\} = -\alpha\left(\delta\left(n_d\right) - n_d\delta'\left(n_d\right)\right)\left(\frac{1-\theta}{\theta}\right)\kappa_d y_d \le 0 \tag{A.88}$$

$$\lim_{i\to 0}\left\{\frac{\partial \mathcal{W}}{\partial N_{f,d}}\right\} = \alpha\delta'\left(n_d\right)\left(1-\theta\right)u\left(y_d\right) \ge 0 \tag{A.89}$$

$$\lim_{i\to 0}\left\{\frac{\partial \mathcal{W}}{\partial y_d}\right\} = \alpha\left(1-n_{c,p}\right)\delta\left(n_d\right)\left(\frac{1-\theta}{\theta}\right)\kappa_d \ge 0 \tag{A.90}$$

In the case where $i \to 0$ and where the optimal tax τ_j is levied on entry, note that the CBDC case will be optimal. The first order conditions of $\bar{\mathcal{W}}$ is

$$\frac{\partial \bar{\mathcal{W}}}{\partial \alpha} = 0 \tag{A.91}$$

$$\frac{\partial \bar{\mathcal{W}}}{\partial \theta} = 0 \tag{A.92}$$

Thus any increase in α or θ will increase welfare of the equilibrium without CBDC while keeping the optimal welfare fixed.

In the case of κ_d, the optimal welfare is also decreasing in κ_d as shown below

$$\frac{\partial \bar{\mathcal{W}}}{\partial \kappa_d} = -\left(1-n_{c,p}\right)\delta\left(n_d\right)\bar{y}_d < 0. \tag{A.93}$$

In the limit as $\theta \to 1$, it is clear that $0 > \frac{\partial \mathcal{W}}{\partial \kappa_d} > \frac{\partial \bar{\mathcal{W}}}{\partial \kappa_d}$ and thus an improvement in digital firm productivity increases the welfare gains of introducing a CBDC.